M2PM3 HANDOUT: THE MAXIMUM MODULUS THEOREM

For information only – not examinable.

Theorem (Maximum Modulus Theorem: Local form). If f is holomorphic in $N(a,R)$, and

$$|f(z)| \leq |f(a)| \qquad \forall z \in N(a,R)$$

– then f is constant.

Proof. Fix r, $0 < r < R$. By CIF,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)\ dz}{z-a} \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta}).ire^{i\theta}d\theta}{re^{i\theta}} \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f(a+re^{i\theta})d\theta. \end{aligned}$$

So

$$|f(a)| \leq \frac{1}{2\pi}|\int ...| \leq \frac{1}{2\pi i}\int_0^{2\pi} |f(a+re^{i\theta})|d\theta \leq \frac{1}{2\pi i}\int_0^{2\pi} |f(a)|d\theta = |f(a)|,$$

by hypothesis. So both inequalities are equalities:

$$\int_0^{2\pi} (|f(a)| - |f(a+re^{i\theta})|)d\theta = 0.$$

The integrand is continuous (as f is holomorphic), and ≥ 0 (by hypothesis). So it is $\equiv 0$. So

$$|f(a)| - |f(a+re^{i\theta})| \quad \forall \theta \in [0,2\pi],\ r \in (0,R).$$

So $|f|$ is constant.

If $f = u + iv$, $|f|^2 = u^2 + v^2$ is constant, c^2 say. So (applying $\partial/\partial x$ and $\partial/\partial y$):

$$2uu_x + 2vv_x = 0, \qquad 2uu_y + 2vv_y = 0.$$

Using the Cauchy-Riemann equations,

$$uu_x - vu_y = 0, \qquad uu_y + vu_x = 0.$$

Multiply the first by u, the second by v and add:

$$(u^2+v^2)u_x = 0, \qquad c^2 u_x = 0.$$

If $c = 0$, $f = 0$, constant. If $c \neq 0$, $u_x = 0$. Similarly, $u_y = 0$. So u is constant. Similarly, v is constant. So $f = u + iv$ is constant. //

Thkeorem (Maximum Modulus Theorem). If D is a bounded domain and f is holomorphic on D and continuous on its closure $\overline{D}$ – then $|f|$ attains its maximum on the boundary $\partial D := \overline{D} \setminus D$.

Prof. D is bounded, so $\overline{D}$ is closed and bounded, so is compact (Heine-Borel Thm.). As $|f|$ is continuous on the compact set $\overline{D}$, it attains its supremum M on $\overline{D}$, at a say.

Assume $a \notin \partial D$ (which will give a contradiction). Then $a \in D$, open, so $N(a, R) \subset D$ for some $R > 0$. So $|f|$ attains its maximum on $N(a, R)$ at a. By the above Local Form, f is constant on $N(a, R)$. So by the Identity Theorem, $f \equiv$ constant.

If f is non-constant, this gives the required contradiction, showing $|f|$ attains its maximum on the boundary ∂D. If f is constant, all points are maxima, a trivial case. //

Theorem (Minimum Modulus Theorem). If f is holomorphic and non-constant on a bounded domain D, then $|f|$ attains its minimum either at a zero of f or on the boundary.

Proof. If f has a zero in D, $|f|$ attains its minimum there. If not, apply the Maximum Modulus Theorem to $1/f$.

Theorem (Maximum Modulus Theorem for Harmonic Functions). If D is a bounded domain, u is harmonic in D and continuous on $\overline{D}$, and $u \leq M$ on ∂D: then $u \leq M$ on $\overline{D}$. That is, u attains its maximum on the boundary ∂D.

Proof: similar to the above – omitted.

Applications.

Applications include asymptotics, in particular the *Saddlepoint method* (Riemann, posthumous, 1892) and *Method of steepest descents* (P. DEBYE, 1909). Suppose we have to estimate a line integral, of a holomorphic function f along a curve γ in a bounded region of holomorphy D. We look for a stationary point z_0 of the integrand $f = u + iv$ on γ. As points on γ (closed) are interior points of D (open), u attains its maximum on the boundary. So $z_0 \in \gamma$ is not a maximum. Arguing similarly for $-f$, it is not a minimum, so must be a saddle-point (see Calculus of Several Variables in Real Analysis). The level curves (contours) u constant near z_0 cut the level curves v constant orthogonally, and these are paths of steepest descent (as with contours on an OS map). As in the Deformation Lemma, we may deform γ to such a path of steepest descent. We must refer for further detail to a book or course on Asymptotics. Suffice it to point out here that applications include Stirling's formula for the factorial, or the Gamma function:

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}} \qquad (n \to \infty), \qquad \Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} \qquad (x \to \infty)$$

(James STIRLING (1692-1770) in 1730).