M2PM3 PROBLEMS 2. 18.1.2011

Q1. (i) Plane polar coordinates (r, θ) fail to be unique everywhere in the complex plane. Where does uniqueness fail, and why?
(ii) Similarly for spherical polar coordinates (r, θ, ϕ) on the sphere.

Q2. (i) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \frac{\pi}{a} \qquad (a > 0).$$

(ii) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab(a + b)} \qquad (a, b > 0).$$

Hint: use partial fractions to reduce to (i) if $a \neq b$. Then use this and continuity if $a = b$. We shall return to this example in Ch. III by residue calculus.

Q3. (i) Show that if

$$F(t) := \int_0^{\infty} e^{-x} \cos xt dx,$$

$$F(t) = 1/(1 + t^2).$$

Hint: integrate by parts twice.
(ii) Deduce that

$$\int_{-\infty}^{\infty} e^{ixt}.\frac{1}{2}e^{-|x|}dx = \frac{1}{(1 + t^2)}.$$

[Note the formal similarity of this with

$$\int_{-\infty}^{\infty} \frac{e^{ixt}}{\pi(1 + x^2)}dx = e^{-|t|},$$

which we shall prove in Ch. III – this is an instance of the Fourier Integral Theorem.]

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