M2PM3 PROBLEMS 4. 10.2.2011

Q1. *Triangle Lemma.*

Let Δ be a triangle in $\mathbf{C}$ with perimeter of length L. Show that if z_1, z_2 are points inside or on Δ,

$$|z_1 - z_2| \le L.$$

[This is "obvious", in that it is geometrically clear – the point is that you are asked for a proof. Reason: this is needed in the proof of Cauchy's Theorem for Triangles.]

Q2. *Harmonic conjugates.*

Show that the following functions u are harmonic, and find the corresponding v and $f = u + iv$:
(i) $u(x, y) = x^3 - 3xy^2 - 2y$.
(ii) $u(x, y) = x - xy$.

Q3. Show that as θ increases from 0 to $\pi/2$, $\sin\theta/\theta$ decreases from 1 to $2/\pi$. [We shall need this result from Real Analysis in Chapter III: Applications.]

Q4. *Euler's Beta integral for the Gamma function.*

Recall that $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$ $(x > 0)$. Show that for $x, y > 0$,

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x, y) := \int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}} dv = \int_0^1 u^{x-1}(1-u)^{y-1} du$$

($B(x, y)$ is called the *Beta function*). In particular, as $\Gamma(1) = 0! = 1$ and the LHS is symmetrical in x and y:

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{(1+v)} dv \qquad (0 < x < 1).$$

[We shall use this in Ch. III to prove $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$.]
Suggested method. Write $\Gamma(x)\Gamma(y)$ as a product of integrals over $(0, \infty)$, in t and u say. Change integration variables to v and w, where $u = tv$, $t(1+v) = w$, and interchange the order of integration.
Alternative method (for those who have met the Gamma density $\Gamma(\lambda)$ in Probability Theory – density $f(x) := x^{\lambda-1}e^{-x}/\Gamma(\lambda)$ for $x > 0$, with parameter $\lambda > 0$, and the convolution formula for densities).

Let X, Y be independent random variables, Gamma distributed with parameters λ, μ. Show that $X + Y$ is also Gamma distributed (use the convolution formula for the density of $X + Y$, and identify the functional form as Gamma). The Beta integral follows on identifying the constant (a density must integrate to 1).

NHB