m2pm3prob5(11).tex

M2PM3 PROBLEMS 5. 19.2.2011

Q1 (*Lagrange's identity*). Show that for $z_1, \ldots, z_n$, $w_1, \ldots, w_n$ complex,

$$\sum_1^n |z_i|^2 \sum_1^n |w_j|^2 - |\sum_1^n z_i w_i|^2 = \sum_{1 \le i < j \le n} |z_i \bar{w}_j - z_j \bar{w}_i|^2.$$

Deduce the Cauchy-Schwarz inequality

$$|\sum_1^n z_i w_i| \le \sqrt{\sum_1^n |z_i|^2} \sqrt{\sum_1^n |w_j|^2}.$$

Q2 (*Weierstrass t-substitution*). If $t := \tan \frac{1}{2}\theta$, show that

$$d\theta = \frac{2dt}{1+t^2}, \qquad \sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \tan\theta = \frac{2t}{1-t^2}.$$

Show that for $-1 < c < 1$,

$$\int_0^\pi \frac{d\theta}{1 + c\cos\theta} = \frac{\pi}{\sqrt{1-c^2}}.$$

(We will meet this example early in Ch. III using complex methods – residue calculus).

Q3. For $C(0,1)$ the unit circle, show that

$$\int_{C(0,1)} cosec^2 z dz = 0.$$

Q5. Show that

$$\int_{C(0,1)} (Im\ z)^2 dz = 0.$$

Note. Cauchy's Theorem does not apply in either of Questions 3 or 4 – and you should say why not.

NHB