m2pm3prob6(11).tex

M2PM3 PROBLEMS 6. 3.3.2011

Q1 (Exam 2009, Q2, after proof of the Theorem of the Antiderivative). Let

$$f(z) := \int_{[1,z]} dw/w \qquad (z \in D),$$

where D is the largest star-domain with star-centre 1 for which the above defines f as a convergent integral.
(a) Find D.
(b) Show that if $g(z) := e^z$, $h(z) := f(g(z))$, then

$$h'(z) = 1, \qquad h(z) = z.$$

Q2. (Exam 2009, Q3, after proof of the Cauchy-Taylor theorem). For a real, define $\binom{a}{n} := a(a-1)\ldots.(a-n+1)/n!$ Show that

$$(1+z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n \qquad (|z| < 1).$$

Check that the radius of convergence of the power series on the right is indeed 1.

Q3. We say that $f(z)$ has a property *at infinity* if $f(1/z)$ has the property at 0. Show that if f is holomorphic in the extended complex plane $\mathbf{C}^*$ (i.e., entire – holomorphic in $\mathbf{C}$ – and holomorphic at ∞), f is constant.

So a non-constant entire function has a singularity at ∞. Give some examples.

Q4. If f is entire and $f(z) = O(|z|^k)$ as $|z| \to \infty$, show that f is a polynomial of degree $\leq k$.

Q5. By considering $\int_\gamma dz/z$ with γ the ellipse $x^2/a^2 + y^2/b^2 = 1$ (a, $b > 0$) parametrized by $x = a\cos\theta$, $y = b\sin\theta$ and using CIF, or otherwise, show that

$$\int_0^{2\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{2\pi}{ab}.$$

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