m2pm3prob7(11).tex

M2PM3 PROBLEMS 7. 10.3.2011

Q1 (from Exam 2010 Q2). For complex variables t, z and integer n, the function $J_n(z)$ is defined as the Laurent coefficient of t^n in the following Laurent expansion:

$$\exp\Big(\frac{1}{2}z\big(t-\frac{1}{t}\big)\Big) = \sum_{n=-\infty}^{\infty} t^n J_n(z).$$

Show that
(i)

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)d\theta;$$

(ii)

$$J_n(z) = \sum_{m=0}^{\infty} \frac{(-)^m(\frac{1}{2}z)^{n+2m}}{m!(n+m)!};$$

(iii) for complex variables y, z,

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y)J_{n-m}(z);$$

(iv) $J_{-n}(z) = (-)^n J_n(z)$;
(v) $|J_n(z)| \leq 1$ for z real.
(The $J_n(z)$ are the *Bessel functions* of order n (Bessel functions $J_\nu(z)$ of non-integer order can also be defined).)

Q2. Show that for $a > b > 0$,

$$I := \int_0^{2\pi} \frac{\sin^2\theta\ d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}(a - \sqrt{a^2 - b^2}).$$

Q3. Find

$$I := \int_0^{\infty} \frac{x^2}{x^4 + 5x^2 + 6}dx.$$

NHB