M2PM3 SOLUTIONS 2. 25.1.2011

Q1. (i) At the origin $r = 0$, the argument θ is not defined uniquely – it can be anywhere in $[0, 2\pi]$.
(ii) In spherical polars (r, θ, ϕ), r is distance from the origin, and the angles are latitude and longitude (actually, colatitude and longitude). At the North Pole, longitude is not uniquely defined – any way you look, you are facing South.
Note. 1. This is connected with the special role of the North Pole in stereographic projections.
2. Near the North Pole, the Earth's surface is approximately flat, and one can use plane polar coordinates as local coordinates. Then non-uniqueness in (ii) reduces to non-uniqueness in (i).

Q2. (i) Put $x = ay$:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = \int_{-\infty}^{\infty} \frac{dy}{a^2(1 + y^2)} = \frac{1}{a}[\tan^{-1} y]_{\infty}^{\infty} = \frac{1}{a}(\pi/2 - (-\pi/2)) = \pi/a.$$

(ii)

$$\frac{1}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{(b^2 - a^2)}\Big(\frac{1}{(x^2 + a^2)} - \frac{1}{(x^2 + b^2)}\Big) \qquad (a \neq b).$$

This integrates to

$$\frac{1}{(b^2 - a^2)}\Big(\frac{\pi}{a} - \frac{\pi}{b}\Big) = \frac{\pi}{ab}\frac{(b - a)}{(b^2 - a^2)} = \frac{\pi}{ab(a + b)}.$$

If $a = b$: let $b \to a$ in the above. The integral $\to \pi/(a^2.2a) = \pi/(2a^3)$. So the answer holds for $a = b$ also. (We shall return to this example later as an application of Cauchy's Residue Theorem. We note its real-variable proof now.)

Q3. (i)

$$\begin{aligned}
F(t) &= \int_0^{\infty} e^{-x} \cos xt dx = -\int_0^{\infty} \cos xt de^{x} \\
&= -[\cos xt.e^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x}(-t \sin xt)dx \\
&= 1 = t\int_0^{\infty} \sin xt de^{-x} \\
&= 1 + t[\sin xt.e^{-x}]_0^{\infty} - t\int_0^{\infty} e^{-x}.t \cos xt dx \\
&= 1 - t^2 \int_0^{\infty} e^{-x} \cos xt dx = 1 - t^2 F(t) :
\end{aligned}$$

$$F(t)(1 + t^2) = 1, \qquad F(t) = 1/(1 + t^2).$$

(ii)

$$\int_{-\infty}^{\infty} e^{ixt}.\frac{1}{2}e^{-|x|}dx = \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx + i\int_{-\infty}^{\infty} \sin xt.\frac{1}{2}e^{-|x|}dx$$

$$= \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx = 1/(1+t^2),$$

by (i) (the second integral is zero: *odd* integrand, symmetric limits. The first integral is twice $\int_0^\infty$: *even* integrand, symmetric limits.

Note. 1. Again, we will return to this later in a complex setting, but note this real-variable proof now.

2. In probabilistic language, this finds the characteristic function of the *symmetric exponential* probability density $\frac{1}{2}e^{-|x|}$ as $1/(1+t^2)$.

NHB