m2pm3soln3.tex

M2PM3 SOLUTIONS 3. 1.2.2011

Q1. Writing c, s, t for $\cos\theta$, $\sin\theta$, $\tan\theta$, then as in lectures, taking real and imaginary parts in de Moivre's theorem for $e^{in\theta}$ gives

$$\cos n\theta = c^n - \binom{n}{2} c^{n-2} s^2 + \binom{n}{4} s^4 \ldots = c^n - \binom{n}{2} c^{n-2}(1-c^2) + \binom{n}{4} c^{n-4}(1-c^2)^2 \ldots = T_n(c),$$

$$\sin n\theta = \binom{n}{1} c^{n-1} s - \binom{n}{3} c^{n-3} s^3 + \binom{n}{5} c^{n-5} s^5 \ldots$$

$$= s[\binom{n}{1} c^{n-1} - \binom{n}{3} c^{n-3}(1-c^2) + \binom{n}{5} c^{n-5}(1-c^2)^2 \ldots] = U_{n-1}(c)s,$$

as required.

Q2. The leading coefficient in T_n is

$$1 + \binom{n}{2} + \binom{n}{4} + \ldots = \sum_{k \ even} \binom{n}{k}.$$

By the Binomial Theorem,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Take $a = b = 1$:

$$2^n = \sum_{k \ even} \binom{n}{k} + \sum_{k \ odd} \binom{n}{k} = \sum_e + \sum_o,$$

say. Take $a = 1$, $b = -1$:

$$0 = \sum_e - \sum_o.$$

Add and halve:

$$2^{n-1} = \sum_e :$$

T_n has leading coefficient 2^{n-1}. Subtract and halve:

$$2^{n-1} = \sum_o :$$

U_{n-1} has leading coeffient 2^{n-1} also.

Q3.

$$T_3(x) = x^3 - \binom{3}{2} x(1-x^2) = x^3 - 3x + 3x^3 = 4x^3 - 3x;$$

$$T_4(x) = x^4 - \binom{4}{2} x^2(1-x^2) + (1-x^2)^2 = x^4 - 6x^2 + 6x^4 + 1 - 2x^2 + x^4 = 8x^4 - 8x^2 + 1;$$

$$\begin{aligned} T_5(x) &= x^5 - \binom{5}{2} x^3(1-x^2) + \binom{5}{4} x(1-x^2)^2 \\ &= x^5 - 10x^3 + 10x^5 + 5x - 10x^3 + 5x^5 \\ &= 16x^5 - 20x^3 + 5x. \end{aligned}$$

Q4. From Q1,

$$\begin{aligned} \tan n\theta &= \frac{\sin n\theta}{\cos n\theta} \\ &= \frac{s[\binom{n}{1}c^{n-1} - \binom{n}{3}c^{n-3}s^2 + \binom{n}{5}c^{n-5}s^4 \ldots]}{c^n - \binom{n}{2}c^{n-2}s^2 + \binom{n}{4}c^{n-4}s^4 \ldots} \\ &= \frac{t[\binom{n}{1} - \binom{n}{3}t^2 + \binom{n}{5}t^4 \ldots]}{1 - \binom{n}{2}t^2 + \binom{n}{4}t^4 \ldots}, \end{aligned}$$

dividing top and bottom by c^n. This is a rational function in t, as required. //

Q5. $\tan 7\theta = 0$ where $t = \tan\theta$ is a root of the numerator in Q4 with $n = 7$, i.e. where

$$t[7 - \binom{7}{3} t^2 + \binom{7}{5} t^4 - t^6] = 0. \qquad (*)$$

But also $\tan 7\theta = 0$ where $\sin 7\theta = 0$, i.e. where $7\theta = n\pi$, i.e. where $\theta = 0, \pi/7, 2\pi/7, \ldots, 6\pi/7$. But $\theta = 0$ corresponds to the factor t in $(*)$, so the six roots of [...] $= 0$ in $(*)$ are $\tan\pi/7, \tan 2\pi/7, \ldots, \tan 6\pi/7$ [2010 Exam, Q1(ii); see link to last year's course on website].

NHB