m2pm3soln4(11).tex

M2PM3 SOLUTIONS 4. 17.2.2011

Q1. *Triangle Lemma.*

Draw the line joining z_1 and z_2, and produce it until it meets triangle Δ – at points Z_1, Z_2 say. Then

$$|z_1 - z_2| \le |Z_1 - Z_2|,$$

with equality iff both z_1, z_2 are *on* Δ rather than inside it (so $z_1 = Z_1$, $z_2 = Z_2$). There are two cases.

(i) Z_1, Z_2 lie on different sides of the triangle. Let Z_3 be the vertex in which these sides meet. Then by the Triangle Inequality,

$$|Z_1 - Z_2| \le |Z_1 - Z_3| + |Z_2 - Z_3| \le L_1 + L_2 \le L,$$

where L_1, L_2 are the lengths of the sides containing Z_1, Z_2. Combining, $|z_1 - z_2| \le L$.

(ii) Z_1, Z_2 lie on the same side, of length L_{12} say. Then

$$|Z_1 - Z_2| \le L_{12} \le L,$$

and the result follows as in (i).

Q2. *Harmonic conjugates.*

(i) For $u = x^3 - 3xy^2 - 2y$: $u_x = 3x^2 - 3y^2$, $u_{xx} = 6x$; $u_y = -6xy - 2$, $u_{yy} = -6x$; $u_{xx} + u_{yy} = 6x - 6x = 0$. So u is harmonic.

$v_y = u_x = 3x^2 - 3y^2$. Integrate wrt y: $v = 3x^2 y - y^3 + F(x)$. Differentiate wrt x: $v_x = -u_y = 6xy + 2 = 6xy + F'(x)$. So $F'(x) = 2$, $F(x) = 2x + c$ (w.l.og. take $c = 0$). So $v = 3x^2 y - y^3 + 2$; $f = u + iv = x^3 - 3xy^2 - 2y + 3ix^2 y - iy^3 + 2ix = (x + iy)^3 + 2i(x + iy)$: $f(z) = z^3 + 2iz$.

(ii) For $u = x - xy$, $u_{xx} = 0$, $u_{yy} = 0$, so u is harmonic.

$v_y = u_x = 1 - y$. Integrate wrt y: $v = y - y^2/2 + F(x)$. Differentiate wrt x: $v_x = F'(x) = -u_y = x$, $F(x) = x^2/2$, $v = y - y^2/2 + x^2/2$; $f = u + iv = x - xy + iy + ix^2/2 - iy^2/2 = (x + iy) + \frac{1}{2}i(x + iy)^2$: $f = z + iz^2/2$.

Q3. Let $f(\theta) := \sin\theta/\theta$. By L'Hospital's Rule, $f(0) = 1$.

$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2}.$$

The denominator is positive. So it suffices to show that the numerator, $g(\theta)$ say, is negative on $(0, \pi/2)$. But

$$g'(\theta) = \cos\theta - \theta\sin\theta - \cos\theta = -\theta\sin\theta < 0 \qquad (0 < \theta < \pi),$$

as required.

Q4. $\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt. \int_0^\infty u^{y-1}e^{-u}du$.
Putting $u = tv$, this gives

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt. \int_0^\infty t^{y-1}e^{-tv}v^{y-1}.tdv,$$

or changing the order of integration and writing $w := t(1+v)$,

$$\int_0^\infty v^{y-1}dv \int_0^\infty t^{x+y-1}e^{-t(1+v)}dv = \int_0^\infty w^{x+y-1}e^{-w}dw. \int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv.$$

As the first integral on RHS is $\Gamma(x+y)$, this gives

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv,$$

giving the first part. For the second part, make the change of variable $u := 1/(1+v)$; then $1-u = v/(1+v)$, $du = -dv/(1+v)^2$, and $v = 0, \infty$ correspond to $u = 1, 0$. So

$$\int_0^\infty \frac{v^{y-1}}{(1+v)^{x+y}}dv = \int_0^1 (1-u)^{x-1}u^{y-1}du.$$

But the LHS, and so the RHS, is symmetrical between x and y, so this completes the proof.

Using the alternative probabilistic method via the convolution formula (we switch from x, y to λ, μ, for convenience): if X, Y have densities f, g, $X+Y$ has density $h = f * g$, the convolution of f and g, where

$$h(x) = \int_0^x f(y)g(x-y)dy \qquad (x > 0).$$

Here $f(x) = x^{\lambda-1}e^{-x}/\Gamma(\lambda)$, $g(x) = x^{\mu-1}e^{-x}/\Gamma(\mu)$, so

$$h(x) = \int_0^\infty f(x-y)g(y)dy = \int_0^\infty \frac{(x-y)^{\lambda-1}e^{-(x-y)}}{\Gamma(\lambda)}.\frac{y^{\mu-1}e^{-y}}{\Gamma(\mu)}dy.$$

The RHS is

$$\frac{e^{-x}}{\Gamma(\lambda)\Gamma(\mu)}.\int_0^x (x-y)^{\lambda-1}y^{\mu-1}dy.$$

Putting $y = xu$ in the integral, this is

$$\frac{x^{\lambda+\mu-1}e^{-x}}{\Gamma(\lambda)\Gamma(\mu)}.\int_0^1 (1-u)^{\lambda-1}u^{\mu-1}du = \frac{x^{\lambda+\mu-1}e^{-x}}{\Gamma(\lambda)\Gamma(\mu)}.B(x,y).$$

This is $c.x^{\lambda+\mu-1}e^{-x}$ for some constant c. This shows two things:
(i) h is a Gamma density, $\Gamma(\lambda+\mu)$, from its functional form,
(ii) $c = 1/\Gamma(\lambda+\mu)$ (this is the constant required to make the density integrate to 1, as it must).
The result follows on equating the two expressions for the constant c.

NHB