m2pm3soln5(11).tex

M2PM3 SOLUTIONS 5. 26.2.2011

Q1.
$$\sum_1^n |z_i|^2 \sum_1^n |w_j|^2 - |\sum_1^n z_i w_i|^2 = LHS_1 - LHS_2,$$
say, where
$$LHS_1 = \sum_i \sum_j z_i \bar{z}_i w_j \bar{w}_j = \sum_k + \sum_{i<j} + \sum_{j<i},$$
say (writing k for $i = j$),
$$LHS_2 = [\sum_i z_i w_i][\sum_j \bar{z}_j \bar{w}_j] = \sum_{i,j} = \sum_k + \sum_{i<j} + \sum_{j<i},$$
say (again with k for $i = j$). Subtracting, the first terms cancel, giving
$$LHS_1 - LHS_2 = \sum_{i<j} [z_i \bar{z}_i w_j \bar{w}_j - z_i w_i \bar{z}_j \bar{w}_j] + \sum_{j<i} [z_i \bar{z}_i w_j \bar{w}_j - z_i w_i \bar{z}_j \bar{w}_j].$$
The right-hand side is
$$RHS = \sum_{i<j} [z_i \bar{w}_j - z_j \bar{w}_i][\bar{z}_i w_j - \bar{z}_j w_i] = \sum_{i<j} [z_i \bar{z}_i w_j \bar{w}_j - z_i \bar{z}_j w_i \bar{w}_j - z_j \bar{z}_i w_j \bar{w}_i + z_j \bar{z}_j w_i \bar{w}_i].$$
The first two terms match the sum $\sum_{i<j}$ above. The other two terms match the $\sum_{j<i}$ term above on interchanging i and j. //

The RHS in Lagrange's identity is non-negative, giving $LHS_2 \leq LHS_1$. This gives the Cauchy-Schwarz inequality on taking square roots. //

Q2. As $t = \tan \frac{1}{2}\theta$,
$$dt = \frac{1}{2}\sec^2 \frac{1}{2}\theta = \frac{1}{2}(1 + \tan^2 \frac{1}{2}\theta) d\theta = \frac{1}{2}(1 + t^2) d\theta : \qquad d\theta = \frac{2dt}{1 + t^2}.$$
By the double-angle formula,
$$\sin\theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta = 2 \tan \frac{1}{2}\theta / \sec^2 \frac{1}{2}\theta = \frac{2t}{1 + \tan^2 \frac{1}{2}\theta} = \frac{2t}{1 + t^2}.$$
So
$$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{4t^2}{(1+t^2)^2} = \frac{1 + 2t^2 + t^4 - 4t^2}{(1+t^2)^2} = \frac{1 - 2t^2 + t^4}{(1+t^2)^2} = \frac{(1-t^2)^2}{(1+t^2)^2} : \quad \cos\theta = \frac{1 - t^2}{1 + t^2}.$$
Then
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2t}{1 - t^2}.$$
For $-1 < c < 1$ and
$$I := \int_0^\pi \frac{d\theta}{1 + c\cos\theta},$$

use the t-substitution. The limits 0 and π for θ correspond to 0 and ∞ for t. Substituting for $d\theta$ and $\cos\theta$ as above and multiplying top and bottom by $(1+t^2)$ gives

$$I = 2\int_0^\infty \frac{dt}{(1+c)+(1-c)t^2}.$$

Substituting $x = t\sqrt{(1-c)/(1+c)}$ gives

$$I = \frac{2}{1+c}\int_0^\infty \frac{dx\sqrt{\frac{1+c}{1-c}}}{1+x^2},$$

or

$$I = \frac{2}{\sqrt{1+c}\sqrt{1-c}}\int_0^\infty \frac{dx}{1+x^2} = \frac{2}{\sqrt{1-c^2}}.[\tan^{-1}x]_0^\infty = \frac{2}{\sqrt{1-c^2}}.\frac{\pi}{2}:$$

$I = \pi/\sqrt{1-c^2}$. //

Q3. $d(\cot z)/dz = cosec^2 z$. So as the unit circle is closed,

$$\int_{C(0,1)} cosec^2 z dz = \int_{C(0,1)} \frac{d}{dz}\cot z dz = \int_{C(0,1)} d\cot z = [\cot z]_{C(0,1)} = 0,$$

by the Fundamental Theorem of Calculus. Cauchy's Theorem does *not* apply, as $cosec^2 z$ has a singularity at 0 (a double pole). [Cauchy's Residue Theorem does apply (the residue is 0 as the pole is double rather than single) – but this comes later.]

Q4. Parametrize $C(0,1)$ by $e^{i\theta}$, $0 \le \theta \le 2\pi$. For $f(z) = (Im\ z)^2$, $z = e^{i\theta}$, $f(z) = \sin^2\theta$, so the integral is

$$I = \int_0^{2\pi} \sin^2\theta.ie^{i\theta}d\theta = -\int_0^{2\pi}\sin^3\theta d\theta + i\int_0^{2\pi}\cos\theta\sin^2\theta dt = I_1 + iI_2,$$

say.

$$I_1 = \int_0^{2\pi}(1-\cos^2\theta)d\cos\theta = [\cos\theta - \frac{1}{3}\cos^3\theta]_0^{2\pi} = 0,$$

by periodicity of cos. Similarly,

$$I_2 = \int_0^{2\pi}\sin^2\theta d\sin\theta = \frac{1}{3}[\sin^2\theta]_0^{2\pi} = 0.$$

Cauchy's theorem does not apply, as $z \mapsto Im\ z$ is not holomorphic.

NHB