m2pm3soln6(11).tex

**M2PM3 SOLUTIONS 6. 10.3.2011**

Q1 (part of Exam 2009 Q2). (a) $D$ is the complex plane with a cut along the negative real axis. For, the formula fails to give a convergent integral for $z = 0$, as the real integral $\int_0^1 dx/x$ diverges. So 0 and all points on the far side of 1 from it, i.e. the negative real axis, are not in $D$. But any point $z$ not in $D$ can be joined to 1 by a line-segment avoiding the singularity at 0; on $[1, z]$ the integrand $1/w$ is continuous, so the integral $\int_{[1,z]} dw/w$ is convergent.
(b) $h'(z) = f'(g(z))g'(z)$ (chain rule) $= g'(z)/g(z)$ ($f'(z) = 1/z$, by the Theorem of the Primitive) $= g(z)/g(z) = 1$ ($g' = g$ for the exponential function $g$). So $h'(z) = 1$, $h(z) = z$ as required.
(Interpretation: $f$ is the inverse function of the exponential function $g$, i.e. the logarithm. The cut serves to make the logarithm single-valued.)
*Note.* If $f$ is holomorphic in a neighbourhood of some point $z$, then $f$ is differentiable in some disc containing $z$, and discs are star-shaped. So the Theorem of the Primitive will always apply *locally* – if we keep the domain small enough. The danger is that if the domain becomes too big, the primitive ceases to be single-valued (and so no longer counts as a function) – as with the logarithm in this case, if the cut is omitted.

Q2 (part of Exam 2009 Q3). Take $f(z) = (1 + z)^a$. The complex power $z^a = \exp(a \log\ z)$ has a branch-point at 0 (as the logarithm does). So $(1 + z)^a$ has a branch-point at $-1$. So the largest disc centre 0 in which it is holomorphic is $D = \{z : |z| < 1\}$. $f'(z) = a(1+z)^{a-1}$, $f''(z) = a(a-1)(1+z)^{a-2}$, ... $f^{(n)}(z) = a(a-1)\dots(a-n+1)(1+z)^{a-n}$. So $f^{(n)}(0)/n! = a(a-1)\dots(a-n+1)/n! = \binom{a}{n}$. By the Cauchy-Taylor theorem,

$$(1 + z)^a = \sum_{n=0}^{\infty} \binom{a}{n} z^n, \qquad |z| < 1.$$

(Newton's General Binomial Theorem). The ratio of consecutive terms on the right is

$$\frac{a(a-1)\dots(a-n)z^{n+1}}{(n+1)!} / \frac{a(a-1)\dots(a-n+1)z^n}{n!} = (a-n)z/(n+1) \to -z \qquad (n \to \infty).$$

By the Ratio Test, this converges where $|-z| < 1$, i.e. where $|z| < 1$, showing directly that the power series on the right has radius of convergence 1.

Q3 (*Liouville's theorem on* $\mathbf{C}^*$). As $f$ is holomorphic for all $|z| \geq 1$ (including $+\infty$), $f(1/z)$ is holomorphic, and so continuous, in the closed unit disc $\bar{D} := \{z : |z| \leq 1\}$. So as $\bar{D}$ is compact, $f(1/z)$ is bounded on $\bar{D}$: $|f(1/z)| \leq M_1$, say, for $|z| \leq 1$, or $|f(z)| \leq M_1$ for $|z| \geq 1$. Similarly, as $f(z)$ is holomorphic, so continuous, in $\bar{D}$, $f$ is bounded on $\bar{D}$: $|f(z)| \leq M_2$, say, for $|z| \leq 1$. So if $M := \max(M_1, M_2)$, $|f(z)| \leq M$ for all $z$ in $\mathbf{C}$: $f$ is bounded. As $f$ is also holomorphic in $\mathbf{C}$, so entire, $f$ is constant, by Liouville's theorem.

So if $f$ is entire and non-constant, $f$ has a singularity at $\infty$.

Examples:

polynomials (non-constant – of degree $\geq 1$);

exponentials ($e^z$, $e^{z^2}$, etc.);

trig functions ($\sin z$, $\cos z$), etc.

Q4.

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma(a,R)} \frac{f(z)}{(z-a)^{n+1}} dz.$$

If $|f(z)| \leq M|z|^k$ for large $|z|$, since on $\gamma(a,R)$, $|z| \leq |a| + R$,

$$|f^{(n)}(a)| \leq \frac{n!}{2\pi} \cdot \frac{M(|a|+R)^k . 2\pi R}{R^{n+1}} = O(1/R^{n-k}) \to 0 \quad (R \to \infty) \quad \text{if } n > k.$$

So $f^{(n)}(a) = 0$ for all $a$, i.e. $f^{(n)} \equiv 0$, i.e. $f$ is a polynomial of degree $\leq k$.

Q5. With $\gamma$ the ellipse $x^2/a^2 + y^2/b^2 = 1$ parametrized by $x = a\cos\theta$, $y = b\sin\theta$, $\int_\gamma dz/z = 2\pi i$, by CIF with $f \equiv 1$, $a = 0$ (or by Cauchy's Residue Theorem when we meet it, since $1/z$ has residue 1 at 0). So as $z = a\cos\theta + ib\sin\theta$ gives $dz = (-a\sin\theta + ib\cos\theta)d\theta$,

$$2\pi i = \int_0^{2\pi} \frac{-a\sin\theta + ib\cos\theta}{a\cos\theta + ib\sin\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)(a\cos\theta - ib\sin\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(b^2 - a^2)\sin\theta\cos\theta + iab(\cos^2\theta + \sin^2\theta)}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta.$$

Equating imaginary parts,

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2\theta + b^2\sin^2\theta} d\theta = 2\pi,$$

whence the result on dividing by $ab$.

NHB