m2pm3soln7(11).tex

M2PM3 SOLUTIONS 7. 17.3.2011

Q1. (i) Take γ the unit circle. Write the generating function as $f(t) = \exp(z(t - 1/t)/2)$, for fixed z. Then Laurent's formula for c_n gives

$$J_n(z) = \frac{1}{2\pi i}\int_\gamma \exp\{z(w - 1/w)/2\}dw/w^{n+1}.$$

With $w = e^{i\theta}$,

$$\begin{aligned} J_n(z) &= \frac{1}{2\pi}\int_0^{2\pi} \exp\{z(e^{i\theta} - e^{-i\theta})/2\}.e^{-(n+1)i\theta}.e^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\{iz(e^{i\theta} - e^{-i\theta})/2i\}.e^{-ni\theta}d\theta \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \exp\{-ni\theta + iz\sin\theta\}d\theta.$$

The RHS is $\int_{-\pi}^{\pi} = \int_{-\pi}^{0} + \int_0^{\pi}$. In the first, replace θ by $-\theta$:

$$J_n(z) = \frac{1}{2\pi}[\int_0^{\pi} \exp\{in\theta - iz\sin\theta\}d\theta + \int_0^{\pi} \exp\{-in\theta + iz\sin\theta\}d\theta] :$$

$$J_n(z) = \frac{1}{\pi}[\int_0^{\pi} \cos\{n\theta - z\sin\theta\}d\theta].$$

(ii)

$$\exp(\frac{1}{2}z(t-1/t)) = \big(\sum_{r=0}^{\infty}(z/2)^r t^r/r!\big).\big(\sum_{s=0}^{\infty}(-)^s(z/2)^s t^{-s}/s!\big) = \sum_{n=-\infty}^{\infty} t^n \sum_{r-s=n}(-)^s(z/2)^{r+s}/r!s!,$$

the rearrangements being justified by the absolute convergence of the exponential series. By uniqueness of Laurent expansions, the coefficient of t^n on RHS is $J_n(z)$. The result follows on replacing s by m, r by $s + n = m + n$.
(iii) $\exp(\frac{1}{2}(y + z)(t - 1/t))$ is both $\sum_n t^n J_n(y + z)$ and
$\exp(\frac{1}{2}y(t-1/t)).\exp(\frac{1}{2}z(t-1/t)) = \sum_r t^r J_r(y).\sum_s t^s J_s(z)$, which is $\sum_n t^n \sum_{r+s=n} J_r(y)J_s(z)$.
Write r, s as m, $n-m$ and equate coefficients: $J_n(y+z) = \sum_m J_m(y)J_{n-m}(z)$.
(iv) This follows from the defining generating function on replacing t by $-t$.

Q2. With $z := e^{i\theta}$ and γ the unit circle,

$$I = \int_\gamma \frac{\left(\frac{z-z^{-1}}{2i}\right)^2 \frac{dz}{iz}}{a + \frac{b}{2}\left(z + z^{-1}\right)} = \frac{i}{2b}\int_\gamma \frac{(z^2 - 1)^2\ dz}{z^2(z^2 + (2a/b)z + 1} = \frac{i}{2b}\int_\gamma F(z)\ dz,$$

say. The roots of the denominator, α and β say, have product 1 (so $\beta = 1/\alpha$), and are given by

$$-\frac{a}{b} \pm \frac{1}{b}\sqrt{a^2 - b^2}.$$

The $-$ gives the root of larger modulus, which is outside γ; we want the root of smaller modulus, inside γ, given by $+$: α say. Then

$$F(z) = \frac{(z^2 - 1)^2}{z^2(z - \alpha)(z - \beta)}.$$

By the Cover-Up Rule (or direct expansion),

$$Res_\alpha F = \frac{(a^2 - 1)^2}{\alpha^2(\alpha - \beta)} = \frac{(\alpha - 1/\alpha)^2}{(\alpha - \beta)} = \frac{(\alpha - \beta)^2}{\alpha - \beta} = \alpha - \beta = 2\sqrt{a^2 - b^2}/b.$$

Expanding $F(z)$ about $z = 0$ gives

$$F(z) = z^{-2}(1 + (2a/b)z + z^2)^{-1}(1 - z^2)^2 = z^{-2}(1 - (2a/b)z + O(z^2)).$$

So picking out the coefficient of z^{-1} (the residue), $Res_0 F = -2a/b$. So by CRT,

$$I = \frac{i}{2b}.2\pi i.(-\frac{2a}{b} + \frac{2\sqrt{a^2 - b^2}}{b}) : \qquad I = \frac{2\pi}{b^2}(a - \sqrt{a^2 - b^2}).$$

Q3. As $x^4 + 5x^2 + 6 = (x^2 + 3)(x^2 + 2)$, which has simple zeros at $\pm i\sqrt{2}$, $\pm i\sqrt{3}$, use $f(z) := z^2/(z^2 + 3)(z^2 + 2)$ and the contour Γ consisting of a large semicircle in the upper half-plane with base $[-R, R]$. Then f has simple poles inside Γ at $i\sqrt{3}$, $i\sqrt{2}$. By Jordan's Lemma, the integral round the semicircle tends to 0 as $R \to \infty$, while the integral along the base tends to $2I$ by symmetry. So by Cauchy's Residue Theorem, $I = \sum Res f$, the sum being over the poles at $i\sqrt{3}$ and $i\sqrt{2}$ inside Γ. As both poles are simple, we can use the Cover-Up Rule:

$$Res_{i\sqrt{3}} f = (i\sqrt{3})^2/[(-3 + 2)(i\sqrt{3} + i\sqrt{3})] = (-3)/[-2i\sqrt{3}] = -i\sqrt{3}/2;$$

$$Res_{i\sqrt{2}} f = (i\sqrt{2})^2/[(-2 + 3)(i\sqrt{2} + i\sqrt{2})] = (-2)/[2i\sqrt{2}] = i\sqrt{2}/2.$$

So by CRT,

$$2I = 2\pi i \sum Res f = 2\pi i.(-)i(\sqrt{3} - \sqrt{2})/2 = \pi(\sqrt{3} - \sqrt{2}) : \quad I = \pi(\sqrt{3} - \sqrt{2})/2.$$