m2pm3prob8(11).tex

M2PM3 SOLUTIONS 8. 24.3.2011

Q1. Use $f(z) = (e^{ipz} - e^{iqz})/z^2$. This has a pole at 0 (apparently double, but actually single: the numerator has a simple zero at 0). In the upper half-plane, we use a semicircular contour, with a small semi-circular indentation to avoid this pole – a contour Γ consisting of:
(i) Γ_1, the line segment $[-R, -r]$ (R large, $r > 0$ small);
(ii) Γ_2, the semi-circle centre 0 radius r, clockwise (-ve sense);
(iii) $\Gamma_3 = [r, R]$;
(iv) Γ_4, the semi-circle centre 0 radius R, anticlockwise (+ve sense).
On Γ_4, $|e^{ipz}| = |e^{ip(x+iy)}| = e^{-py} \leq 1$, as $p \geq 0$ and $y \geq 0$ in the upper half-plane, and similarly $|e^{iqz}| \leq 1$. So $|f(z)| = O(1/R^2)$, and by (ML),
$\int_{\Gamma_4} f = O(1/R^2).\pi R = O(1/R) \to 0$ as $R \to \infty$.
As $R \to \infty, r \to 0$, $\int_{\Gamma_1} f + \int_{\Gamma_3} f \to I$, the required integral. On Γ_2, $z = re^{i\theta}$,

$$f(z) = [(1+ipz-p^2z^2/2...)-(1+iqz-q^2z^2/2...)]/z^2 = [i(p-q)-\frac{1}{2}(p^2-q^2)z+...]/z,$$

$dz/z = ire^{i\theta}d\theta/re^{i\theta} = id\theta$, where θ goes from π to 0. So (changing the sign to interchange the limits of integration)

$$\int_{\Gamma_2} f = -\int_0^\pi [i(p-q) + O(r)](id\theta) \to \pi(p-q) \qquad (r \to 0).$$

Since $\int_\Gamma f = 0$ by Cauchy's Theorem, this gives $I + \pi(p-q) = 0$: $I = -\pi(p-q)$.

Q2. Use $f(z) = 1/(z^4 + a^4)$ and as contour γ the interval $\gamma_1 := [-R, R]$, completed by a semi-circle γ_2 of radius R in the upper half-plane. On γ_2, $|f| = O(1/R^4)$, so by ML $\int_{\gamma_2} f = O(1/R^3) \to 0$ as $R \to \infty$, while $\int_{\gamma_1} f \to 2I$ by symmetry. The integrand f has poles where $z^4 = -a^4 = a^4e^{i\pi}$, $z = ae^{i\pi/4}$, $ae^{3i\pi/4}$, $ae^{5i\pi/4}$, $ae^{7i\pi/4}$; only the first two matter (are inside γ). If α is such a root $\alpha^4 = -a^4$, and we can evaluate the residue at α as

$$Res_\alpha f = \lim_{z \to \alpha} \frac{z - \alpha}{z^4 - \alpha^4}.$$

As $z^4 - \alpha^4 = (z-\alpha)(z^3 + z^2\alpha + z\alpha^2 + \alpha^3)$, the RHS is

$$1/(z^3 + z^2\alpha + z\alpha^2 + \alpha^3) \to 1/(4\alpha^3) = \alpha/(4\alpha^4) = -\alpha/(4a^4) \qquad (z \to \alpha)$$

(by the Cover-Up Rule, or directly). So by CRT,

$$2I = 2\pi i.\frac{-1}{4a^4}(ae^{i\pi/4} + ae^{3i\pi/4}).$$

But

$$e^{i\pi/4} + e^{3i\pi/4} = e^{i\pi/4}(1 + e^{i\pi/2}) = \frac{1}{\sqrt{2}}(1+i).(1+i) = 2i/\sqrt{2} = i\sqrt{2}.$$

So $I = \sqrt{2}\pi/(4a^3)$. //

Q3. Put $f(z) = (\pi \cot \pi z)/(1 + z + z^2)$. Since $z^3 - 1 = (z - 1)(z^2 + z + 1)$, the roots of $z^2 + z + 1$ are $e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ and $e^{4\pi i/3} = -1/2 - i\sqrt{3}/2$, the complex cube roots of unity other than 1. Integrating f round the square contour Γ_n with vertices $(n + 1/2)(\pm 1 \pm i)$ gives

$$\int_{\Gamma_n} f = 2\pi i \Big(\sum_{k=-n}^{n} \frac{1}{1 + k + k^2} + Res_{e^{2\pi i/3}} f + Res_{e^{4\pi i/3}} f \Big).$$

$$\int_{\Gamma_n} f = O(1/n^2).O(n) = O(1/n) \to 0 \qquad (n \to \infty) \qquad \text{(by ML)}.$$

Combining,

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + n + n^2} = -\Big(Res_{e^{2\pi i/3}} + Res_{e^{4\pi i/3}} \Big) \frac{\pi \cot \pi z}{(z - e^{2\pi i/3})(z - e^{4\pi i/3})}.$$

By the Cover-Up Rule, the RHS is

$$-\frac{\pi \cot(\pi e^{2\pi i/3})}{i\sqrt{3}} + \frac{\pi \cot(\pi e^{4\pi i/3})}{i\sqrt{3}} = \frac{i\pi}{\sqrt{3}}[(\cot(-\frac{\pi}{2} + \frac{i\pi\sqrt{3}}{2}) - \cot(-\frac{\pi}{2} - \frac{i\pi\sqrt{3}}{2})].$$

Since $\tan(a + \pi/2) = -\cot a$, $\cot(a - \pi/2) = -\tan a$, the RHS is

$$\frac{i\pi}{\sqrt{3}}[\tan(-\frac{i\pi\sqrt{3}}{2}) - \tan(\frac{i\pi\sqrt{3}}{2})] = \frac{2i\pi}{\sqrt{3}} \tan(-\frac{i\pi\sqrt{3}}{2}).$$

As $i \tan i\theta = \tanh \theta$, this is $(2i\pi/\sqrt{3}).(-i).\tanh(\pi\sqrt{3}/2)$. So

$$\sum_{n=-\infty}^{\infty} \frac{1}{1 + n + n^2} = \frac{2\pi}{\sqrt{3}} \tanh(\pi\sqrt{3}/2).$$

Q4. For $m > 0$, put $u := mx$. Since $dx/x = du/u$, this reduces the problem to the case $m = 1$, which gives $I = \pi/2$ (Lectures). For $m < 0$, we get $I = -\pi/2$, since the integrand is odd in m. For $m = 0$, we get 0 since the integrand is 0.

NHB