SQUARES LEMMA

For information only – not examinable.

SQUARES LEMMA. For C_N the square contours with vertices $(N+\frac{1}{2})(\pm 1 \pm i)$, the functions $cosec\, \pi z$, $\cot \pi z$ are uniformly bounded (in z and N) on the C_N.

Proof. On the horizontal sides, $z = x + iy$, $|y| \geq 1/2$. Then

$$|cosec\ \pi z| = 1/(\frac{1}{2}|e^{i\pi z} - e^{-i\pi z}|).$$

Now $|e^{i\pi z}| = |e^{i\pi x}.e^{-\pi y}| = e^{-\pi y}$, $|e^{-i\pi z}| = e^{\pi y}$, and as $|z_1 - z_2| \geq ||z_1| - |z_2||$, $1/|z_1 - z_2| \leq 1/||z_1| - |z_2||$. So

$$|cosec\ \pi z| \leq 1/(\frac{1}{2}|e^{-\pi y} - e^{\pi y}|).$$

The RHS is $1/(\frac{1}{2}|e^{\pi y} - e^{-\pi y}|)$ if $y \geq 0$, $1/(\frac{1}{2}|e^{-\pi y} - e^{\pi y}|)$ if $y \leq 0$. So RHS $= 1/sh|\pi y|$. But $|y| \geq 1/2$, $sh' = ch > 0$, so $sh \uparrow$. So $1/sh \downarrow$, so RHS $\leq 1/sh(\pi/2)$.

Similarly, $cot = cos/sin = cos\ cosec$,

$$|\cos \pi z| = \frac{1}{2}|e^{i\pi z} = e^{-i\pi z}| \leq \frac{1}{2}(|e^{i\pi z} + |e^{-i\pi z}| = \frac{1}{2}(e^{-\pi y} + e^{\pi y}) = ch\ \pi y.$$

So

$$|cot\ \pi z| = |\cos \pi z||cosec\ \pi z| \leq ch\ \pi y/sh\ \pi|y| = coth\ \pi|y| \leq coth(\pi/2),$$

as $|y \geq 1/2$, and $coth \downarrow$ (check!).

On the vertical sides, $z = \pm(N + \frac{1}{2}) = iy$ ($|y| \leq N + \frac{1}{2}$), so

$$\begin{aligned} |cosec\ \pi z| = 1/|\sin \pi z| &= 1/|\sin(\pm\pi(N + \frac{1}{2}) + i\pi y)| \\ &= 1/|\cos(i\pi y)| \quad \text{(trig addition formulae)} \\ &= 1/ch|\pi y| \\ &\leq 1, \end{aligned}$$

as $ch \uparrow$ on $\mathbf{R}$. Similarly, the trig addition formulae used again give

$$|cot\ \pi z| = \frac{|\pm \sin i\pi y|}{|\pm \cos i\pi y|} = |\tan i\pi y| = |1 - e^{-2\pi y}|/|1 + e^{2\pi y}| \leq 1.$$

Combining gives the result. //

Cor. $cosec\ z$, $cot\ z$ are uniformly bounded on the squares Γ_N with vertices $(N + \frac{1}{2})\pi(\pm 1 \pm i)$.

NHB