m3pm16cwsoln.tex

SOLUTIONS TO ASSESSED COURSEWORK 2012

Q1 (L, §155 p. 588-590).
Th. $\psi(x) \sim x$ (i.e. PNT) $\Rightarrow N(x) := \sum_{n \le x} \mu(n) \log n = o(x \log x)$.
Proof. Consider the Dirichlet series of $-\zeta'(s)/\zeta^2(s)$ in two ways:
(i)

$$\frac{1}{\zeta(s)} . \frac{-\zeta'(s)}{\zeta(s)} = (\sum \mu(n)/n^s)(\sum \Lambda(n)/n^s) = \sum (\mu * \Lambda)(n)/n^s,$$

$$(\mu * \Lambda)(n) = \sum_{d|n} \Lambda(d)\mu(n/d).$$

(ii)

$$\frac{d}{ds}\Big(\frac{1}{\zeta(s)}\Big) = -\frac{\zeta'(s)}{\zeta^2(s)} : \quad -\frac{\zeta'(s)}{\zeta^2(s)} = \frac{d}{ds}(\sum \mu(n)/n^s) = -\sum \mu(n) \log n/n^s.$$

Equating coefficients,

$$\mu(n) \log n = -\sum_{d|n} \Lambda(d)\mu(n/d). \qquad [\mathbf{4}]$$

So

$$N(x) := \sum_{n \le x} \mu(n) \log n = -\sum_{n \le x} \sum_{d|n} \Lambda(d)\mu(n/d) = -\sum_{jk \le x} \Lambda(j)\mu(k)$$

$$= -\sum_{k \le x} \sum_{j \le x/k} \Lambda(j) = -\sum_{k \le x} \mu(k)\psi(x/k).$$

As $\psi(x) \sim x$, given, for all $\epsilon > 0$ there exists $m = m(\epsilon)$ s.t.

$$|\psi(x) - x| < \epsilon x \qquad (x \ge m). \qquad (1)$$

Split the sum for $N(x)$ into sums for $k \le x/m$ and $x/m < k \le x$: $|N(x)| \le |\sum ...| + |\sum ...| = \sum_1 + \sum_2$, say.

By II.5, Prop.,

$$|\sum_{n \le y} \mu(n)/n| \le 1 \qquad \text{for all } y. \qquad (2)$$

So

$$\begin{aligned}\sum\nolimits_1 &= |\sum_{k\le x/m}\mu(k)\psi(x/k) \le |\sum\{\psi(x/k)-x/k\}\mu(k)| + |\sum(x/k)\mu(k)| \\ &< \epsilon\sum_{k\le x/m} x/k + x \qquad \text{(by (1) and (2))} \\ &< \epsilon x(\log(x/m)+1)+x \qquad \text{(by I.4)} \\ &< \epsilon x\log x + x. \qquad\qquad \textbf{[3]}\end{aligned}$$

As $|\mu(.)| \le 1$, $\psi(.) \ge 0$ and $\psi(.)$ is increasing,

$$\sum\nolimits_2 = |\sum_{x/m<k\le x}\mu(k)\psi(x/k)| \le \sum_{x/m<k\le x}\psi(x/k) \le \psi(m)\sum_{x/m<k\le x}1 \le \psi(m).x.$$

Combining, $N(x)| \le \epsilon x\log x + x + x\psi(m)$, giving $N(x) = o(x\log x)$. // **[3]**

Q2 (L, §155, p.588-590).
Th. $\psi(x) \sim x$ (i.e. PNT) $\Rightarrow M(x) := \sum_{n\le x}\mu(n) = o(x)$.

Proof. We use Q1. As $N(x) = \sum_{n\le x}\mu(n)\log n$, $\mu(n)\log n = N(n) - N(n-1)$, $\mu(n) = (N(n)-N(n-1))/\log n$. So

$$M(x) = \sum_{n\le x}\mu(n) = 1 + \sum_{2\le n\le x}\frac{N(n)-N(n-1)}{\log n}$$

$$= 1 + \sum_{2\le n\le x} N(x)\Big(\frac{1}{\log n} - \frac{1}{\log(n+1)}\Big) + \frac{N([x])}{\log([x]+1)}, \qquad \textbf{[4]}$$

by partial summation. But

$$\Big(\frac{1}{\log n} - \frac{1}{\log(n+1)}\Big) = \frac{\log(1+1/n)}{\log n\log(n+1)} < \frac{1}{n}.\frac{1}{\log n\log(n+1)} < \frac{1}{n\log^2 n}. \quad \textbf{[2]}$$

So by Q1,

$$M(x) = 1 + o\Big(\sum_{2\le n\le x}\frac{n\log n}{n\log^2 n}\Big) + o\Big(\frac{x\log x}{\log x}\Big) = 1 + o(\sum_{2\le n\le x}1/\log n) + o(x). \quad \textbf{[2]}$$

By I.4, the sum here is of order $\int_2^x dt/\log t = li(x) \sim x/\log x$, giving $M(x) = o(x)$. // **[2]**

NHB