m3pm16eandpi.tex

e **AND** π

e.

The number

$$e + \sum_{n=0}^{\infty} 1/n!, \qquad (e)$$

the base of natural logarithms, was introduced by Sir Isaac NEWTON (1642-1727) in 1665. It received its modern name, e, from Euler in 1727. Euler also derived the *continued fraction* (HW, Ch. X) for e in 1737:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\ddots}}}}},$$

or in more compact notation

$$e = 2 + \frac{1}{1+}\,\frac{2}{2+}\,\frac{3}{3+}\cdots$$

He also proved that a number has a *finite* continued fraction expansion iff it is *rational* (HW, Th. 161). So:

Cor. (Euler). e is irrational.

For another proof, write $I_1 := [2, 3]$. Inductively, divide I_{n-1} $(n = 2, 3, \ldots)$ into equal subintervals, the second of which is I_n. Thus

$$I_2 = [5/2!, 6/2!], \qquad I_3 = [16/3!, 17/3!], \qquad I_4 = [65/4!, 66/4!], \ldots$$

Then

$$\bigcap_{n=1}^{\infty} I_n = \{e\}$$

expresses (e). For $n > 1$, $I_{n+1} \subset I_n^o = (a_n/n!, (a_n + 1)/n!)$, say ($A^o$ is the interior of A, as in M2PM3). So the point of intersection is not a fraction with denominator $1/n!$ for any $n \geq 1$. But any rational p/q with $q > 0$ is of this form:

$$\frac{p}{q} = \frac{p.(q-1)!}{q!}.$$

So e is irrational. //

A number is *algebraic* if it is the root of a polynomial equation with integer coefficients ('surd-like': e.g. $\sqrt{2}$, a root of $x^2 - 2 = 0$), *transcendental* otherwise. The number of polynomials of degree n with integer coefficients is countable for each n; the union of countably many countable sets is countable; so the number of algebraic numbers is countable. But the reals are uncountable. So (i) transcendental numbers exist; (ii) most reals are transcendental – indeed, almost all reals are transcendental (both in the sense of ordinary language and in the sense of Measure Theory).

One would expect e to be transcendental. It is (Charles HERMITE (1822-1901) in 1873).

π.

From the point of view of Number Theory, the natural way to expand π is also as a continued fraction:

$$\pi/4 = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{\ddots}}}} = \frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots$$

(William, Lord BROUNCKER (1620-1684), in 1655 – related to Wallis' product for π, for which see e.g. M2PM3, L32). As this continued fraction is infinite, one has as above:

Cor. π is irrational.

Again as above, one would expect π to be transcendental. It is (C. L. F. LINDEMANN (1852-1939) in 1882).

The theory of transcendental numbers is very important and interesting, but we cannot pursue it here. See e.g.

Serge LANG, *Introduction to transcendental numbers*, Addison-Wesley, 1966.

Brouncker's proof of the continued fration for π was long thought lost. It has recently been reconstructed; see

S. KRUSHCHEV, *Orthogonal polynomials and continued fractions, from Euler's point of view*, CUP, 2008, Ch. 3.