m3pm16enh.tex

M4PM16 ENHANCED COURSEWORK 2012

Set Th. 22 March 2012 (Week 10); deadline 12 noon, Th. 3 April

Q1. By considering the Dirichlet series of both sides of the identity

$$\zeta(s) + \frac{\zeta'(s)}{\zeta(s)} - 2\gamma = \frac{1}{\zeta(s)}(\zeta^2(s) + \zeta'(s) - 2\gamma\zeta(s))$$

and equating coefficients, or otherwise, show that if

$$b(n) := d(n) - \log n - 2\gamma,$$

then

$$B(x) := \sum_{n \le x} b_n = O(\sqrt{x}).$$

(You may assume any formula for $\sum_{n \le x} d_n$, and any form of Stirling's formula, but state any such result you use clearly and give a detailed reference.)

Q2. Assume $M(x) = o(x)$. By using Dirichlet's Hyperbola Identity (DHI) for the sum function

$$A(x) := \sum_{n \le x} a_n,$$

where $a_n := 1 - \Lambda(n)$ for $n \ge 2$ and $a_1 := 1 - 2\gamma$, or otherwise, show that

$$A(x) = o(x).$$

Deduce that $\psi(x) \sim x$, i.e. PNT holds, and thus that PNT is equivalent to $M(x) = o(x)$.
(DHI gives three terms, one sum – $\sum_1 := \sum_y$ say – with $B(.)$ in the summand, another sum, $\sum_2 := \sum_z$, with $M(.)$ in the summand. Choose $\epsilon > 0$ arbitrarily small, and then use Q1 to take z so large that $|\sum_1| < \epsilon x$. For this fixed z, use the assumption $M(x)/x = o(1)$ termwise in $\sum_2$. Estimate the product term as with $\sum_1$.)

NHB