SOLUTIONS TO ENHANCED COURSEWORK 2012

Q1. By II.9 on DHI,

$$\sum_{n \le x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Also

$$\sum_{n \le x} \log n = \log([x]!).$$

By Stirling's formula (see e.g.[WW], §12.33)

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}}, \qquad \log n! \sim -n + (n + \frac{1}{2}) \log n = n \log n - n + O(\log n).$$

So

$$\sum_{n \le x} b(n) = \sum_{n \le x} d(n) - \sum_{n \le x} \log n - 2\gamma \sum_{n \le x} 1$$

$$= x \log x + (2\gamma - 1)x + O(\sqrt{x}) - [x \log x - x] - 2\gamma x = O(\sqrt{x}). \qquad \textbf{[6]}$$

Q2 (Landau, 1912; see e.g. G. H. HARDY, *Divergent Series*, OUP, 1949, Appendix IV, 378 footnote and 379-380). On the left, we have

$$\zeta(s) + \frac{\zeta'(s)}{\zeta(s)} - 2\gamma = \sum_1^\infty \frac{1 - \Lambda(n)}{n^s} - 2\gamma = \sum_1^\infty a_n / n^s,$$

where $a_1 := 1 - 2\gamma$ and $a_n := 1 - \Lambda(n)$ for $n \ge 2$. But on the right, we have the product of two Dirichlet series, with coefficients $\mu(n)$, $d(n) - \log n - 2\gamma$. So the right is a Dirichlet convolution, and equating coefficients gives

$$1 - \Lambda(n) = \sum_{jk=n} \mu(j)[d(k) - \log k - 2\gamma] = \sum_{jk=n} \mu(j) b(k)$$

for $n \ge 2$. So

$$x - \psi(x) = \sum_{n \le x} (1 - \Lambda(n)) = \sum_{jk \le x} \mu(j) b(k). \qquad \textbf{[6]}$$

By DHI, with $yz = x$ the hyperbola,

$$A(x) := \sum_{n \le x} a_n = \sum_{jk \le x} \mu(j) b(k) = \sum_{j \le y} \mu(j) B(x/j) + \sum_{k \le z} b(k) M(x/k) - M(y) B(x/y)$$

$$= \sum_1 + \sum_2 + \sum_3,$$

say. As $|\mu(.)| \le 1$ and $|B(.)| = O(\sqrt{.})$,

$$|\sum_1| = O(\sum_{j \le y} \sqrt{x/j}) = O(\sqrt{x} \sum_{j \le y} 1/\sqrt{j}) = O(\sqrt{x}.\sqrt{y}) \le M\sqrt{x}\sqrt{y} = Mx/\sqrt{z},$$

say. With (y_1, z_1) some point on the hyperbola, $y_1 = x/z_1$: $|\sum_1| \le Mx/\sqrt{z_1}$. Given $\epsilon > 0$ arbitrarily small, choose z_1 so large that $M/\sqrt{z_1} < \epsilon$, and fix it:

$$|\sum_1| < \epsilon x. \qquad (i)$$

As the sum in $\sum_2$ is finite, and in each term $M(x/k) = o(x/k) = o(x)$, $\sum_2 = o(x)$. *(ii)*

Also $|\sum_3| = |M(y_1)|.|B(z_1)| = O(M(y_1)) = O(M(x/z_1)) = o(x/z_1) = o(x)$. *(iii)*

Combining (i), (ii) and (iii) gives

$$A(x) = o(x).$$

But this and

$$A(x) := \sum_{n \le x} a_n = const + \sum_{n \le x} (1 - \Lambda(n)) = const + [x] - \psi(x) = O(1) + x - \psi(x)$$

give $\psi(x) = x + o(x)$. So $\psi(x) \sim x$, PNT: $M(x) = o(x)$ implies PNT. // **[6]**

In Assessed Coursework Q2, we showed that PNT implies $M(x) = o(x)$. Combined with the above, this shows that $M(x) = o(x)$ is equivalent to PNT in the form $\psi(x) \sim x$. **[2]**

Note. The result above is due to Landau in 1912 (paper in the *Wiener Sitzungsberichte*, reprinted in his *Collected Papers*). As Hardy remarks in his obituary of Landau (JLMS 13 (1938), 302-310; *Collected Papers* VII), Landau refused to write a second edition of his path-breaking *Handbuch* of 1909, although his own later work had superceded parts of it. As Hardy further remarks in the quoted Appendix of *Divergent Series* in 1949, the argument had not then appeared in any book. Subsequent progress has changed the perspective on these things: Jameson, in his *PNT*, our recommended text, derives various results equivalent to PNT as different special cases of a common theorem (see J §3.4, and our III.8).

NHB