m3pm16l10.tex
Lecture 10. 7.2.2012

Cor. $\zeta(s) \neq 0$ for $Res > 1$.

Proof. This is clear from the product, but not from the series!

Prop. (Euler). $\sum_{p \leq x} 1/p \geq \log\log x - \frac{1}{2}$. In particular, $\sum_p 1/p$ diverges.

Proof. With $\sum_x^*$ a sum over all n with all prime factors $\leq x$,

$$T_x := \prod_p 1/(1-1/p) = \sum\nolimits_x^* 1/n \geq \sum\nolimits_1^x 1/n > \log x.$$

But for $0 < x < 1$

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots < x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots = x + \frac{\frac{1}{2}x^2}{1-x},$$

so

$$-\log(1-1/y) - 1/y < \frac{1}{2y^2(1-1/y)} = \frac{1}{2y(y-1)}.$$

So if $S_x := \sum_{p \leq x} 1/p$,

$$\log T_x - S_x = \sum_{p \leq x}\Big(-\log(1-\frac{1}{p}) - \frac{1}{p}\Big) < \frac{1}{2}\sum_{p \leq x}\frac{1}{p(p-1)}$$

$$< \frac{1}{2}\sum_2^\infty \frac{1}{n(n-1)} = \frac{1}{2}\sum_2^\infty\Big(\frac{1}{n-1} - \frac{1}{n}\Big) = \frac{1}{2}$$

(the sum telescopes). So

$$S_x \geq \log T_x - \frac{1}{2} \geq \log\log x - \frac{1}{2}. \qquad //$$

Cor. (Euclid). There are infinitely many primes (I.1).

§5. THE MÖBIUS FUNCTION

The *Möbius function* μ is defined by
$\mu(1) := 1$;

$\mu(n) := (-)^k$ if n is a product of distinct primes;
$\mu(n) := 0$ if n contains a square factor (equivalently, a square prime factor p^2).

Theorem. If a is completely multiplicative with $\sum |a_n| < \infty$,

$$\frac{1}{\sum_1^\infty a_n} = \sum_1^\infty \mu(n) a_n.$$

Proof. As $|\mu(.)| \leq 1$, $\sum \mu(n) a_n$ converges. If its sum is S, and

$$Q_N := \prod_{p \leq N} (1 - a_p) :$$

multiply out on RHS. Each n which is a product of k distinct primes each $\leq N$ contributes $(-)^k a_n$. No n containing a square does. So (notation as above)

$$Q_N = \sum\nolimits_{n \in E_N} \mu(n) a_n.$$

So

$$|S - Q_N| \leq \sum\nolimits_{n \in E_N^*} |a_n| \leq \sum\nolimits_{n > N} |a_n| \to 0 \qquad (N \to \infty). \qquad //$$

Cor. (i)

$$\frac{1}{\zeta(s)} = \sum\nolimits_1^\infty \mu(n)/n^s \qquad (Re\ s > 1); \qquad (\mu, 1/\zeta)$$

(ii)

$$u * \mu = e_1; \qquad \sum\nolimits_{i|n} \mu(i) = 0 \qquad (n > 1).$$

Proof. (i) Take $a_n = 1/n^s$ in the Theorem.
(ii) Use the identity

$$1 = \zeta(s).1/\zeta(s) : \qquad 1 \leftrightarrow e_1, \zeta \leftrightarrow u, 1/\zeta \leftrightarrow \mu. \qquad //$$

For completeness only, we add the following self-contained proof of (ii). For $n = 1$, $u(1)\mu(1) = 1.1 = 1$; for $n > 1$, $(u * \mu)(n) := \sum_{i|n} \mu(i)$. If $n = p_1^{r_1} \dots p_k^{r_k}$ (from FTA), the $i > 1$ with $\mu(i) \neq 0$ are of the form $i = q_1 \dots q_j$ with the qs distinct primes from $\{p_1, \dots, p_k\}$. There are $\binom{k}{j}$ such choices, each giving an i with $\mu(i) = (-)^j$. As $\binom{n}{0} = 1$, this holds also for $j = 0$. So by the Binomial Theorem,

$$(u * \mu)(n) = \sum_{i|n} \mu(i) = \sum_{j=0}^k (-)^j \binom{k}{j} = (1 - 1)^k = 0. \qquad //$$