## m3pm16l12.tex Lecture 12. 9.2.2012

 $\log \zeta(s)$ 

**Theorem.** Write  $c_n := 1/m$  if  $n = p^m$  is a prime power, 0 otherwise. Then

$$H(s) := \sum_{1}^{\infty} c_n / n^s = \log \zeta(s) \qquad (Re \ s > 1)$$

*Proof.*  $\zeta(s) = \prod 1/(1-p^{-s})$ , so  $\log(\zeta(s)) = -\sum_p \log(1-p^{-s})$ . As  $-\log(1-z) = \sum_1^\infty z^m/m$  for |z| < 1,

$$\log \zeta(s) = \sum_{p} \sum_{1}^{\infty} \frac{1}{m} \cdot 1/p^{ms} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=p^{m}} 1/n^{s} = \sum_{n=1}^{\infty} c_{n}/n^{s} = H(s),$$

in the half-plane of absolute convergence  $Re \ s > 1$ , where changing of order of summation is justified. The function  $\log \zeta(s)$  is holomorphic on  $\sigma > 1$  as  $\zeta(s)$  is holomorphic and nonzero there. //

## The Von Mangoldt Function $\Lambda(n)$ , 1899.

 $\Lambda(n):=\log(p)$  if  $n=p^m$  is a prime power, 0 otherwise. Differentiating the result above gives

$$\zeta'(s)/\zeta(s) = -\sum_{1}^{\infty} c_n \log n/n^s.$$

But

 $c_n \log n = \frac{1}{m} \cdot m \log p = \log p$  if  $n = p^m$  is a prime power , 0 otherwise.

So

$$-\zeta'(s)/\zeta(s) = \sum_{1}^{\infty} \Lambda(n)/n^s \qquad (Re \ s > 1). \qquad (\Lambda, -\zeta'/\zeta)$$

**Corollary**. With  $l(n) := \log(n)$ ,  $\Lambda * u = l$ ,  $l * \mu = \Lambda$ .

Proof.  $\Lambda(1) = l(1) = 0$ . For n > 1,  $n = p_1^{r_1} \dots p_k^{r_k}$ , say. Then  $(\Lambda * u)(n) = \sum_{i|n} \Lambda(i)$ . The divisors *i* of *n* are  $i = p_1^{s_1} \dots p_1^{s_k}, 0 \le s_j \le r_j$ . Those with  $\Lambda(i) \neq 0$  are only those with  $i = p_j^{s_j}$ , each of which has  $\Lambda(i) = \log p_j$ . There

are k elements in this sum (ignoring the case of i = 1).  $\sum_{i|n} \Lambda(i) = \sum_{j=1}^{k} r_j \log p_j = \log \prod_j p_j^{r_j} = \log n = l(n)$ . So  $\Lambda * u = l$ , and then  $l * \mu = \Lambda$  follows by Möbius inversion. //

Write

$$\psi(x) := \sum_{n \le x} \Lambda(n).$$

Then also

$$\psi(x) = \sum_{p^m \le x} \log p.$$

As the highest power m with  $p^m \leq x$  is  $m = \left[\frac{\log x}{\log p}\right]$ :

$$\psi(x) = \sum_{p \le x} \left[ \frac{\log x}{\log p} \right] \log p.$$

By Abel summation (I.3),  $-\zeta'(s)/\zeta(s) = \sum_{1}^{\infty} \Lambda(n)/n^{s}$ , and

$$\frac{\zeta'(s)}{\zeta(s)} = -s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx \qquad (Re \ s > 1).$$

We shall see that PNT  $(\pi(x) \sim li(x) = x/\log x) \Leftrightarrow \psi(x) \sim x$ , and this is the way we will prove the Prime Number Theorem.

## 7. Mertens' Theorems

**Theorem** (HW Th. 424). For  $x > 1, \sum_{n \le x} \Lambda(n)/n = \log x + r(x)$ , with  $|r(\cdot)| \le 2$ .

*Proof.* By II.5,  $\left|\sum_{1}^{N} \mu(n)/n\right| \leq 1$ . Write  $S(x) := \sum_{n \leq x} \log n$ . As  $\Lambda * u = l$ ,

$$S(x) = \sum_{n \le x} (\Lambda * u)(n) = \sum_{n \le x} \Lambda(n) \left[\frac{x}{n}\right] \quad \text{(with [.] the integer part)}$$
$$= x \sum_{n \le x} \frac{\Lambda(n)}{n} - a(x), \quad \text{where} \quad a(x) := \sum_{n \le x} -\Lambda(n) \{x/n\},$$

with  $\{.\}$  the fractional part. Then

$$0 \le a(x) = \sum_{n \le x} \Lambda(u) \{ x/n \} \le \sum_{n \le x} \Lambda(u) = \psi(x) \le 2x,$$

by Chebyshev's Upper Estimate (III.2).