m3pm16l13.tex
Lecture 13. 14.2.2012

Now use the Integral Test argument (I.4) to estimate $S(x)$. As

$$\int_1^x \log t dt = x \log x - x + 1$$

(integrate by parts), this gives

$$S(x) = x \log x - x + b(x), \qquad |b(x)| \leq \log x + 1.$$

Now $\log x + 1 \leq x$ for $x > 1$ (integrate $1/x \leq 1$ from 1 to x). So $|b(x)| \leq x$. So

$$x \sum_{n \leq x} \Lambda(n)/n = S(x) + a(x) = x \log x - x + a(x) + b(x).$$

But $0 \leq a(x) \leq 2x$, $|b(x)| \leq x$, so $|a(x) - x + b(x)| \leq 2x$. //

Cor.

$$\int_1^x \frac{\psi(t)}{t^2} dt = \log x + O(1) \qquad (x > 1).$$

Proof. By Abel summation (I.3),

$$\sum_{n \leq x} \Lambda(n)/n = \frac{\psi(x)}{x} + \int_1^x \frac{\psi(t)}{t^2} dt.$$

But $\psi(x)/x$ is bounded (from Chebyshev's θ-function: III.2), so this follows from the Theorem. //

The next result shows that we can neglect the powers of primes in the Theorem (at the cost of losing the bound 2): *powers of primes become sparse*, so this is not too surprising.

Theorem (MERTENS' FIRST THEOREM: F. MERTENS (1840-1927) in 1874; HW Th. 425).

$$\sum_{p \leq x} \log p/p = \log x + O(1) \qquad (x > 1) \qquad (|O(.)| \leq 4).$$

Proof. As $\Lambda(n) = \log p$ when $n = p^m$,

$$0 \leq \sum_{n \leq x} \Lambda(n)/n - \sum_{p \leq x} \log p/p \leq \sum_{p \leq x} \log p \Big(\frac{1}{p^2} + \frac{1}{p^3} + \ldots \Big)$$

Summing the geometric series, the RHS is

$$\sum_{p \le x} \frac{p}{p(p-1)} \le \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)}.$$

The series converges, to sum ≤ 2 (check). The result then follows from the Theorem above. //

Theorem (MERTENS' SECOND THEOREM; HW Th. 427).

$$\sum_{p \le x} 1/p = \log\log x + C_1 + O(1/\log x) \qquad (x \ge 2),$$

for some constant C_1.

Proof (Compare $\sum_{n \le x} 1/n = \log x + \gamma + o(1)$, I.4). We use Abel summation, with

$$a(n) := \log n/n \qquad (n \text{ prime}), \qquad 0 \text{ otherwise}, \qquad A(x) := \sum_{n \le x} a_n.$$

By Mertens' First Theorem,

$$A(x) = \log x + r(x), \qquad |r(.)| \le c_0 \qquad (x > 1),$$

$a(1) = 0$, and

$$\sum_{p \le x} 1/p = \sum_{2 \le n \le x} \frac{a(n)}{\log n}.$$

By Abel summation, this gives

$$\sum_{p \le x} 1/p = \frac{A(x)}{\log x} + \int_2^x \frac{A(t)}{t \log^2 t} dt = 1 + \frac{r(x)}{\log x} + \int_2^x \frac{dt}{t \log t} + I(x),$$

where

$$I(x) := \int_2^x \frac{r(t)}{t \log^2 t} dt.$$

But

$$\int_2^x \frac{1}{t \log t} dt = \log\log x - \log\log 2,$$

$$\int_2^\infty \frac{dt}{t \log^2 t} < \infty, \qquad \text{as} \qquad \frac{1}{t \log^2 t} = -\frac{d}{dt}\Big(\frac{1}{\log t}\Big).$$

So $I(x) \to I$, finite, as $x \to \infty$, and

$$I(x) = I - s(x), \qquad |s(x)| \le c_0 \int_x^\infty \frac{dt}{t \log^2 t} = \frac{c_0}{\log x}.$$

This gives the result with $C_1 := 1 - \log\log 2 + I$. //