

Theorem(Merten's Formula, HW Th 929).

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x} \quad (x \rightarrow \infty).$$

Proof. Write $\Sigma := \sum_p \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)$, which is convergent. By Merten's Second Theorem and the Constants Lemma (from the Website),

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + C_1 + o(1) = \log \log x + \gamma + \Sigma + o(1).$$

Now,

$$\sum_{p \leq x} \left(\log \left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = \Sigma + o(1),$$

from the definition of Σ . Subtracting:

$$\sum_{p \leq x} \log \left(1 - \frac{1}{p}\right) = -\log \log x - \gamma + o(1).$$

That is,

$$\log \left[\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \right] = \log \left[\frac{e^{-\gamma}}{\log x} \right] + o(1).$$

So

$$\log \left[\prod_{p \leq x} \left(1 - \frac{1}{p}\right) / \frac{e^{-\gamma}}{\log x} \right] \rightarrow 0 \quad (x \rightarrow \infty).$$

So $[\dots] \rightarrow 1$, i.e.

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}. \quad //$$

8. Prime Divisor Functions

Recall the following arithmetic functions: $d(n) := \# \text{divisors of } n$; $\omega(n) := \# \text{ distinct prime divisors } n$; $\Omega(n) := \# \text{ prime divisors } n$ (counted with multiplicity). So if $n = p_1^{r_1} \dots p_k^{r_k}$, we have

$$d(n) = \prod_1^k (1 + r_j), \quad \omega(n) = k, \quad \Omega(n) = \sum_1^k r_j.$$

Theorem. (i) $\sum_{n \leq x} \omega(n) = x \log \log x + C_1 x + O(1/\log x)$,
(ii) $\sum_{n \leq x} \Omega(n) = x \log \log x + C_2 x + O(1/\log x)$,
where as above

$$C_1 = \gamma + \sum_p \frac{1}{p(p-1)}, \quad C_2 = C_1 + \sum_p \frac{1}{p(p-1)} = C_1 + S,$$

say.

Proof. (i) $\sum_{n \leq x} \omega(n)$ is the number of pairs (p, n) with $p|n$ and $n \leq x$. For fixed p , the number of such pairs is equal to the number of multiples $rp \leq x$, i.e. $[x/p]$. So

$$\sum_{n \leq x} \omega(n) = \sum_{p \leq x} \left(\frac{x}{p} - \left\{ \frac{x}{p} \right\} \right).$$

By the first theorem of II.7,

$$\sum_{p \leq x} \frac{x}{p} = x \sum_{p \leq x} \frac{1}{p} = x \log \log x + C_1 x + O(x/\log x).$$

Also

$$0 \leq \sum_{p \leq x} \{x/p\} < \sum_{p \leq x} 1 = \pi(x) = O(x/\log x),$$

by Chebyshev's Upper Estimate (III.2). Combining gives (i).

(ii) Similarly. //

Note. This theorem gives us:

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} \omega(n) &= \log \log x + C_1 + O(1/\log x); \\ \frac{1}{x} \sum_{n \leq x} \Omega(n) &= \log \log x + C_2 + O(1/\log x). \end{aligned}$$

In Probabilistic Number Theory (III.9, III.10.1), we think of the LHS as the 'mean value' of ω, Ω . This says that both $\sim \log \log x$. This result, due to Hardy and Ramanujan (1917), corresponds to the Law of Large Numbers (LLN), thinking of ω, Ω as random and divisibility by distinct primes as independent events. We shall prove the Prime Number Theorem (PNT), and extensions of it, counting $n \leq x$ with k prime factors. This will correspond to the relevant Central Limit Theorem (CLT), extending the Law of Large Numbers.