m3pm16l16.tex
Lecture 16. 21.2.2012

The *Riemann Hypothesis* (RH) of 1859 is that the only zeros of ζ in the critical strip are on the *critical line*

$$\sigma = \frac{1}{2}. \qquad (RH)$$

RH is still open, and is the most famous and important open question in Mathematics. Its resolution would have vast consequences for prime-number theory (especially error terms in PNT – see e.g. J Ch. 5). It is so hard that proving theorems *conditional on RH* (i.e., assuming it is true) is respectable in Analytic Number Theory.

PNT was proved independently in 1896 by J. HADAMARD (1865-1963, French) and Ch. de la Vallée Poussin (1866-1962, Belgian). Both used Complex Analysis and ζ.

Since counting primes relates to $\mathbf{N}$ ($\subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R} \subset \mathbf{C}$), it seemed strange and unaesthetic to use complex methods. Great efforts were make to provide an *elementary proof*, over half a century.

Elementary proofs of PNT were found in 1948 by Paul ERDÖS (1913-1996, Hungarian) and Atle SELBERG (1917-2009, Norwegian). There is a full account by J. Spencer and R. Graham in *The Mathematical Intelligencer*, **31**.3 (2009), 18-23. Erdös gave an elementary proof of a result of Chebyshev (proof of Bertand's postulate: Problems 8). Selberg told him the day after seeing it that he could use it to complete an elementary proof of PNT. Erdös proposed collaboration but Selberg declined; their papers were published separately in 1949.

Proofs of ANT by complex methods are in all the books on ANT, including J Ch. 3, which we follow. Elementary proofs of PNT are harder; see e.g. HW Ch. XXII (22.14-16), J Ch. 6, A Ch. 4, R Ch. 13.

Error estimates in PNT are very important. Naturally, complex methods give better error estimates than elementary ones. Error estimates depend on *zero-free regions* of ζ (to the left of the 1-line, in the critical strip) – the bigger, the better; see III.10.2.

§2. CHEBYSHEV'S THEOREMS

Defn. (CHEBYSHEV, 1850). $\theta(x) := \sum_{p \leq x} \log p$.

So if $p_1, \dots, p_n$ are the primes $\leq x$, $\theta(x) = \log p_1 + \dots + \log p_n = \log(p_1 \dots p_n)$.

Propn. $\theta(x) \leq \pi(x) \log x$.

Proof. Above: $n = \pi(x)$ and each $\log p_j \leq \log x$. //

By Abel summation,

$$\theta(x) = \textstyle\sum_{n \leq x} I_P(n) \log n = \pi(x) \log x - \displaystyle\int_2^x \frac{\pi(y)}{t} dt. \qquad (\theta - \pi)$$

Conversely, π can be expressed in terms of θ. As $\theta(x) := \sum_{n \leq x} b_n$, where $b_n := \log n$ if n is prime, 0 otherwise, and $b_1 = 0$, Abel summation gives

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t} dt \qquad (x \geq 2) \qquad (\pi - \theta)$$

Write

$$li(x) := \int_2^x \frac{dt}{\log t}$$

for the *logarithmic integral* ($li(x) := 0$ for $x \leq 2$). Then by Problems 1,

$$li(x) \sim x/\log x \qquad (x \to \infty),$$

and it turns out that

$$\pi(x) \sim li(x) \qquad (x \to \infty) \qquad (PNT)$$

is a more accurate form of PNT than $\pi(x) \sim x/\log x$.

Theorem 1 (Chebyshev). (i) If $c_0 \leq \theta(x) \leq C_0 x$ ($x \geq 2$), then for $\alpha := 2/\log 2$,

$$c_o(li(x) + \alpha) \leq \pi(x) \leq C_0(li(x) + \alpha) \qquad (x \geq 2).$$

(ii) If $\epsilon > 0$ and $cx \leq \theta(x) \leq Cx$ ($x \geq x_0$), then there exists x_1 such that

$$(c - \epsilon) li(x) \leq \pi(x) \leq (C + \epsilon) li(x) \qquad (x \geq x_1).$$

Proof. As in Problems 1: integrating by parts,

$$li(x) := \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} - \alpha + \int_2^x \frac{dt}{\log^2 t}.$$

Then $(\pi - \theta)$ gives (i). For (ii), split $\int_2^x$ in $(\pi - \theta)$ into $\int_2^{x_0} + \int_{x_0}^x$ and use the upper bound given ($li(x) \to \infty$, so it 'swallows constants'). Similarly for the lower bound. //