m3pm16l2.tex

Lecture 2. 17.1.2012

Theorem (Euclid). There are infinitely many primes.

Proof. Assume not. Then some list $p_1, \ldots, p_n$ exhausts the primes. Consider

$$N := 1 + p_1 p_2 \ldots p_n.$$

Then p_1 does not divide N: N has remainder 1 when divided by p_1. Similarly, $p_2, \ldots, p_n$ do not divide N. So as these are all the primes, N does not contain a prime factor, contradicting FTA. //

2. Limits of Holomorphic Functions

Theorem. Let f_n be holomorphic on a domain D. If $f_n \to f$ uniformly on compact subsets K of D, then f is holomorphic and $f_n^{(k)} \to f^{(k)}$.

Proof. For any contour Γ with Γ and $\mathrm{int}(\Gamma)$ contained in D, let $K := \Gamma \cup \mathrm{int}(\Gamma)$. Then $\int_\Gamma f_n = 0$ by Cauchy's Theorem, and K is compact (Heine-Borel). So by uniformity,

$$0 = \int_\Gamma f_n \to \int_\Gamma f : \qquad \int_\Gamma f = 0,$$

for any choice of Γ, meaning that f is holomorphic (Morera). By Cauchy's integral formula $CIF(k)$,

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{f_n(w)dw}{(w-z)^{k+1}} \to \int_\Gamma \frac{f(w)dw}{(w-z)^{k+1}},$$

by uniform convergence. But the RHS is $f^{(k)}(z)$, by $CIF(k)$ again. //

§3. Abel (= partial) summation

Calculus (differentiation, integration, their links, etc.) used to be called *infinitesimal calculus*. It has a discrete counterpart, the Calculus of Finite Differences (differencing, summing, their links, etc.). This is more basic, and

more messy (because of 'end terms'). It is needed for numerical work (interpolation pre-computers, discretisation post-computers).
Standard notation. Given a sequence $a(1), a(2), \ldots$, write a_n, $a(n)$ interchangeably,

$$A(n),\ A_n := \sum_{k=1}^{n} a_k \qquad (A_0 = 0,\ A_1 = a_1).$$

Similarly for $b(n)$, b_n, B_n etc.

The basic result of calculus is the Fundamental Theorem of Calculus ($\int_a^b F' = F(b) - F(a)$). The discrete analogue of this is *telescoping sums*: sums of differences telescope:

$$(a_1 - a_0) + (a_2 - a_1) + \ldots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = a_n - a_0.$$

Integration by parts,

$$(fg)' = f'g + fg', \quad f(b)g(b) - f(a)g(a) = \int_a^g f'g + \int_a^b fg' : \quad \int_a^b fg' = [fg]_a^b - \int_a^b f'g,$$

has a discrete analogue, Abel/partial summation, below.

Abel's Lemma: For integers $n > m \geq 0$,

$$\sum_{m+1}^{n} a_r f_r = \sum_{m}^{n-1} A_r [f_r - f_{r-1}] + A_n f_n - A_m f_m.$$

Proof:

$$\sum_{m+1}^{n} a_r f_r = (A_{m+1} - A_m) f_{m+1} + \ldots + (A_n - A_{n-1}) f_n$$

$$= -A_m f_{m+1} + A_{m+1}(f_{m+1} - f_{m+2}) + \ldots + A_{n-1}(f_{n-1} - f_n) + A_n f_n \qquad (*)$$

$$= -A_m f_m + A_m(f_m - f_{m+1}) + \ldots + A_{n-1}(f_{n-1} - f_n) + A_n f_n \quad \text{(adding and subtracting } A_m f_m)$$

$$= \sum_{m}^{n-1} A_r [f_r - f_{r+1}] + A_n f_n - A_m f_m. \qquad //$$

Cor. $\sum_1^n f_r = \sum_1^{n-1} r[f_r - f_{r+1}] + n f_n$.

Proof. Take $a_r \equiv 1$, so $A_r = r$, $A_0 = 0$. //