m3pm16l21.tex
Lecture 21. 1.3.2012

5. PERRON'S FORMULA (Oskar Perron (1880-1975) in 1908).

As before, $f(s) := \sum_1^\infty a_n/n^s,\ A(x) := \sum_{n \le x} a_n$.

Write $\int_{c-iT}^{c+iT}$ for a line integral (M2PM3, II.9, L15) along the line $\sigma = c, -T \le t \le T$. If both $\int_c^{c+iT}$ and $\int_{c-iT}^{c}$ have limits as $t \to \infty$, then so does $\int_{c-iT}^{c+iT}$ and we write the limit as $\int_{c-i\infty}^{c+i\infty}$.

Write $E(x) := 1\ (x \ge 1),\ 0\ (x < 1)$ – the *unit step function* or *Heaviside function* (probability distribution function of the constant 1).

Proposition 1. For $x > 0, c > 0$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2} dx = E(x) \log x.$$

Proof. Choose $R > c$, and write $C : C(0, R)$ for the circle centre 0, radius R. $I(C_1 \cup L_R) := \frac{1}{2\pi i} \int_{C_1 \cup L_R} x^s ds/s^2$. Write $\lambda = \log x$. Then

$$\frac{x^s}{s^2} = \frac{e^{\lambda s}}{s^2} = \frac{1}{s^2}\left(1 + \lambda s + \frac{1}{2}\lambda^2 s^2 + ...\right).$$

This function of s is holomorphic inside the contour $L_R \cup C_1$, except for a double pole at $s = 0$, of residue λ. As $|x^s| = x^\sigma$, and $x > 1$, on C_1 we have $|x^s| \le x^c$, so $|x^s/s^2| \le x^c/R^2$. So by the ML Inequality (M2PM3, II.9, L16),

$$|I(C_1)| := \left|\frac{1}{2\pi i} \int_{C_1} \frac{x^s}{s^2}\right| \le \frac{1}{2\pi} \cdot \frac{x^c}{R^2} \cdot 2\pi R = x^c/R \to 0 \qquad (R \to \infty).$$

By Cauchy's Residue Theorem (M2PM3, II.11, L25),
$I(C_1 \cup L_R) = \lambda$, so $I(L_R) \to \lambda = \log x$ as $R \to \infty$.
This is the statement for $x \ge 1$.

For $0 < x < 1$, instead integrate around $L_R \cup C_2$. Now there are no singularities inside the contour, so by CRT $I(C_2 \cup L_R) = 0$. Now $x^\sigma \downarrow$ in x, so $|x^s| \le x^c$ for $s \in C_2$. As before, this gives $I(C_2) \to 0$, so $I(L_R) \to 0$. The same proof, using $\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}$, gives

Proposition 2. For $x > 0,\ c > 1$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s-1)} ds = (x - 1)E(x).$$

Theorem 1 (PERRON'S FORMULA). Suppose that $f(s) := \sum_1^\infty a_n/n^s$ converges absolutely for $\mathrm{Re}(s) > 1$, (i.e $\sigma_a \leq 1$ in II.1), and let $A(x) := \sum_{n \leq x} a_n$. Then for $c > 1, x > 1$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s-1)} f(s) ds = \sum_{n \leq x} a_n \left(\frac{1}{n} - \frac{1}{x} \right) = \int_1^x \frac{A(y)}{y^s} dy.$$

Proof. Write $x^s f(s) = G(s) + H(s)$, $G(s) := \sum_{n \leq x} a_n (x/n)^s$, $H(s) := \sum_{n > x} a_n (x/n)^s$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(s)}{s(s-1)} ds = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n \leq x} a_n \frac{(x/n)^s}{s(s-1)} ds$$

$$= \sum_{n \leq x} a_n \cdot \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s-1)} ds = \sum_{n \leq x} a_n \cdot (\frac{x}{n} - 1),$$

by Proposition 2.

Write $M := \sum_{n > x} |a_n| (x/n)^c$. Note $M < \infty$ by absolute convergence in $\mathrm{Re}(s) > 1$, as $c > 1$. For $\mathrm{Re}(s) \geq c$, as $n > x$, $|(x/n)^s| \leq (x/n)^c$, so $|H(s)| \leq M$. As H is holomorphic, Cauchy's Theorem gives

$$\int_{C_2 \cup L_R} \frac{H(s)}{s(s-1)} ds = 0.$$

As before, $\int_{C_2} \to 0$ as $R \to \infty$. So $\int_{L_R} \to 0$ also:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{H(s)}{s(s-1)} ds = 0.$$

The first statement follows, dividing both sides by x. For the second statement, use Abel Summation (I.3 Cor(ii)) for $f(y) = 1/y$, so $f'(y) = -1/y^2$. //

Theorem 2. If (i) $f(s) := \sum_1^\infty a_n/n^s$ is absolutely convergent for $\mathrm{Re}(s) > 1$;
(ii)

$$f(s) = \frac{\alpha}{s-1} + \alpha_0 + (s-1)h(s),$$

with h is holomorphic at $s = 1$ (so if $\alpha \neq 0$, f has a simple pole at 1 of residue α);
(iii) for $t_0 \geq 1$, $|f(\sigma \pm it)| \leq P(t)$ when $\sigma \geq 1$ and $t \geq t_0$, with $\int_1^\infty \frac{P(t)}{t^2} dt < \infty$
– then

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx \text{ converges to } \alpha_0 - \alpha.$$