m3pm16l22.tex
Lecture 22. 6.3.2012

Proof. Write $\phi(s) := h(s)/s$ (holomorphic at 1); then by (ii)

$$\begin{aligned}
\phi(s) = \frac{h(s)}{s} &= \frac{f(s)}{s(s-1)} - \frac{\alpha}{s(s-1)^2} - \frac{\alpha_0}{s(s-1)} \\
&= \frac{f(s)}{s(s-1)} - \frac{\alpha}{s(s-1)}\Big(\frac{s}{s-1} - 1\Big) \qquad (\frac{1}{s-1} = \frac{s}{s-1} - 1) \\
&= \frac{f(s)}{s(s-1)} - \frac{\alpha}{(s-1)^2} + \frac{\alpha}{s(s-1)} - \frac{\alpha_0}{s(s-1)} \\
&= \frac{f(s)}{s(s-1)} - \frac{\alpha}{(s-1)^2} - \frac{\alpha'}{s(s-1)} \qquad (\alpha' := \alpha_0 - \alpha).
\end{aligned}$$

For $s = \sigma + it$ with $\sigma \geq 1$, $|t| \geq t_0$,

$$|s(s-1)\phi(s)| = |f(s) - \frac{s}{s-1}\alpha - \alpha'| \leq |f(s)| + |\alpha||\frac{s}{s-1}| + |\alpha'|.$$

In the stated region (shaded),

$$|\frac{s}{s-1}| \leq \frac{t_0}{\sqrt{1+t_0^2}}$$

(ratio of distances of s to 0 and 1: maximise this by taking s as close as possible to 1). So by (iii),

$$|s(s-1)\phi(s)| \leq P(t) + |\alpha|.\frac{t_0}{\sqrt{1+t_0^2}} + |\alpha'| = P_1(t), \text{ say, where } \int_1^\infty \frac{P_1(t)}{t^2} dt < \infty.$$

In the shaded region,

$$|s| = \sqrt{\sigma^2 + t^2} \geq |t|, \quad |s-1| = \sqrt{(\sigma-1)^2 + t^2} \geq |t|, \quad \text{so} \quad |s(s-1)| \geq t^2.$$

So

$$|\phi(s)| = \Big|\frac{P_1(t)}{s(s-1)}\Big| \leq \frac{P_1(t)}{t^2}, \quad \text{and} \quad \int_1^\infty \frac{P_1(t)}{t^2} dt < \infty. \qquad (*)$$

For $x > 1$, $c \geq 1$, define

$$I = I(x,c) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s-1} \phi(s) ds.$$

For $c > 1$, expressing ϕ in terms of f gives

$$I(x,c) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{s-1}f(s)ds - \frac{\alpha}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{(s-1)^2}ds - \frac{\alpha'}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s-1)}ds$$

$$= I_1 - I_2 - I_3, \qquad \text{say.}$$

By the Theorem above,

$$I_1 = \int_1^x \frac{A(y)}{y^2}dy.$$

By Proposition 1 above, $I_2 = \alpha \log x$
(recall $x > 1$: $E(x) = 1$). By Proposition 2 above,

$$I_3 = \alpha'(1 - \frac{1}{x}).$$

Combining,

$$I(x,c) = \int_1^x \frac{A(y) - \alpha y}{y^2}dy - \alpha'(1 - \frac{1}{x}). \qquad (**)$$

Lemma. $I(x,c)$ is independent of $c > 1$.

Proof. We show $I(x,c) = I(x,1)$ for $c > 1$.

$$I(x,c) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(c,t)dt, \qquad g(c,t) = x^{c-1+it}\phi(c+it).$$

Choose $\epsilon > 0$, and then $T \geq t_0$ so large that $\int_T^\infty (P_1(t)/t^2)dt < \epsilon$. If $1 \leq c \leq 2$, $|x^{c-1+it}| = x^{c-1} \leq x$, so

$$\int_T^\infty |g(c,t)|dt \leq x\int_T^\infty |\phi(c+it)|dt \leq x\int_T^\infty \frac{P_1(t)}{t^2}dt < \epsilon x,$$

and similarly for $\int_{-\infty}^{-T}$. On the compact set $1 \leq c \leq 2$, $-T \leq t \leq T$, $g(c,t)$ is continuous, so *uniformly continuous* (Heine's Theorem). So for c close enough to 1,

$$|g(c,t) - g(1,t)| \leq \epsilon/2T \qquad \forall \epsilon \in [-T,T],$$

and then

$$|\int_{-T}^T g(c,t)dt - \int_{-T}^T g(1,t)dt| \leq \int_{-T}^T |g(c,t) - g(1,t)|dt \leq \epsilon.$$

Combining,

$$|I(x,c) - I(x,1)| \leq \epsilon(1 + 2x).$$

As x here is fixed, this shows that $I(x,c)$ is independent of c. //