m3pm16l26.tex

Lecture 26. 13.3.2012

Theorem 1 can be applied to extend III.5 Th. 2 and III.6 Th. 2. We give the extension to III.6; III.5 is similar (see J, 133 and 105, 118).

Theorem 2. If (i) $f(s) = \sum_1^\infty a_n/n^s$ converges for $Re\ s > 1$, and f can be continued analytically to a region $\supset \{Re\ s \geq 1\}$ except possibly for $s = 1$,
(ii)

$$f(s) = \frac{\alpha}{s-1} + \alpha_0 + (s-1)h(s), \qquad h \text{ holomorphic at } 1,$$

(iii') $|A(x)| \leq Mx$ $(x \geq 1)$, and *either*
(iv) $a_n \geq 0$ for all n, *or*
(iv') there exist b_n, $g(s) := \sum_1^\infty b_n/n^s$ satisfying (i) and (ii) with $|a_n| \leq b_n$ for all n – then

(a) $A(x)/x \to \alpha$ $(A(x) := \sum_{n \leq x} a_n)$;
(b) $\sum_{n \leq x} a_n/n - \alpha \log x \to \alpha_0$;
(c) $\sum_1^\infty (a_n - \alpha)/n$ converges to $\alpha_0 - \gamma\alpha$ (γ Euler's constant).

Proof. (a) By III.5 Th. 2 or III.6 Th. 2,

$$\int_1^\infty \frac{A(x) - \alpha x}{x^2} dx \qquad \text{converges.}$$

If (iv) holds, the Elementary Tauberian Theorem, Th. 1, gives $A(x)/x \to \alpha$. Similarly for (iv') (if a_n is real, B and $B - A$ are increasing; argue as in (iv); if a_n is complex, apply this to $Re\ a_n$, $Im\ a_n$).
(b) In either case, by Abel summation

$$\sum_{n \leq x} a_n/n - \alpha \log x = \frac{A(x)}{x} + \int_1^x \frac{A(y)}{y^2} dy - \alpha \int_1^x \frac{dy}{y}$$

$$= \frac{A(x)}{x} + \frac{A(y) - \alpha y}{y^2} dy \to \alpha + (\alpha_0 - \alpha) = \alpha_0 \qquad (x \to \infty).$$

(c) So

$$\sum_{n \leq x} \frac{a_n - \alpha}{n} = \sum_{n \leq x} \frac{a_n}{n} - \alpha \log x - \alpha(\sum_{n \leq x} 1/n - \log x) \to \alpha_0 - \gamma\alpha,$$

by (b) and I.4. //

§8. PROOF OF PNT

THEOREM (PNT). (1) $\psi(x) \sim x$.
(2) $\pi(x) \sim li(x) \sim x/\log x$.

Proof. (1) Let $f(s) := -\zeta'(s)/\zeta(s) = \sum_1^\infty \Lambda(n)/n^s$ (II.6). This is holomorphic in $Re\ s > 0$ except at $s = 1$, so Th. 2 of III.7 above holds. Also

$$f(s) = \frac{1}{s-1} - \gamma + (s-1)h(s)$$

with h holomorphic at 1 (II.3), so (ii) of Th. 2 holds with $\alpha = 1$ and $\alpha_0 = -\gamma$. By Chebyshev's Upper Estimate (III.2), $\psi(x) \leq 2x$, so (iii') of Th. 2 holds. As $\Lambda(.) \geq 0$, (iv) of Th. 2 holds.

So by (a) of Th. 2, $\psi(x)/x \to 1$, giving (1).
(2) This is equivalent to (i) by Chebyshev's results (III.2). //

Cor. $\sum_{n \leq x} \Lambda(n)/n - \log x \to -\gamma$;
$\sum_1^\infty (\Lambda(n))/n$ converges to -2γ;

$$\int_1^\infty \frac{\psi(x) - x}{x^2} dx \quad \text{converges to } -\gamma - 1.$$

Proof. The first two follow by (b) and (c) of Th. 2 of III.7. The third follows by Th. 2 of III.6. //

Applying III.7 to $1/\zeta(s)$ rather than to $\zeta'(s)/\zeta(s)$ gives ($M(x) := \sum_{n \leq x} \mu(n)$):
(i) $\int_1^\infty M(x)dx/x^2 = 0$;
(ii) $M(x)/x \to 0$;
(iii) $\sum_1^\infty \mu(n)/n = 0$.
See J 134-5 for details ((iii) is due to von Mangoldt in 1898).

We shall see in III.10.3 that each of (ii) and (iii) is actually *equivalent to PNT*. Similarly, so is $\zeta(1 + it) \neq 0$ (III.4).

Jameson's method derives the above from different special cases of general theorems, in which a complex Tauberian condition (involving analytic continuability, as above) gives conclusions of integral, series and limit type. One can see the distinction from the usual approach (using $\zeta(1+it) \neq 0$, and some form of Wiener's Tauberian theorem – see III.10.3), and the elementary proof of PNT (see e.g. R §13.2), which makes no use of the zeta function.