m3pm16l27.tex
Lecture 27. 15.3.2012

9. LANDAU'S POISSON EXTENSION OF PNT: PRIMES PLAY A GAME OF CHANCE

Theorem (LANDAU 1900; Handbuch, 1909, 203-211). If $\pi_k(x)$ is the number of $n \le x$ with k distinct prime factors ($k = 1, 2, \ldots$),

$$\pi_k(x) \sim \frac{x}{(k-1)!} . \frac{(\log\log x)^{k-1}}{\log x}.$$

Lemma (Handbuch, 203-5). For $F(u, x)$ $(2 \le u \le x)$ s.t.
(i) $F(u, x) \ge 0$;
(ii) for fixed $x > 2$ $F(u, x)/\log u$ decreases in u;
(iii) $F(2, x) = o(\int_2^x F(u, x) du/\log u)$ – then

$$\sum_{p \le x} F(p, x) \sim \int_2^x \frac{F(u, x)}{\log u} du.$$

Proof. By PNT, $\theta(x) \sim x$, so $\theta(x) = x + x\epsilon(x)$, $\epsilon(x) = o(1)$. So

$$\sum_{p \le x} F(p, x) = \sum_{n=2}^{x} \frac{\theta(n) - \theta(n-1)}{\log n} F(n, x) \qquad \text{(definition of } \theta)$$

$$= \sum_2^x \frac{F(n, x)}{\log n} + \sum_2^{x-1} n\epsilon(n)\Big[\frac{F(n, x)}{\log n} - \frac{F(n+1, x)}{\log(n+1)}\Big] + \frac{F(2, x)}{\log 2} + [x]\epsilon([x])\frac{F([x], x)}{\log[x]}, \qquad (i)$$

by Abel summation. As in the Integral Test (I.4),

$$\sum_2^x \frac{F(n, x)}{\log n} + \frac{F(2, x)}{\log 2} = (1 + o(1)) \int_2^x \frac{F(u, x)}{\log u} du.$$

Choose $\epsilon > 0$ arbitrarily small; there exists $U = U(\epsilon)$ with $|\epsilon(u)| < \epsilon$ for $u > U$. So for $x > U + 1$, the sum of the remaining terms on the RHS of (i) is

$$|\sum_2^{n-1} n\epsilon(n)\Big[\frac{F(n, x)}{\log n} - \frac{F(n+1, x)}{\log(n+1)}\Big] + [x]\epsilon([x])\frac{F([x], x)}{\log[x]}|$$

$$< O(F(2,x)) + \epsilon \sum_U^{n-1} [...] + \epsilon[x]F([x],x)/\log[x]$$

$$= \epsilon \sum_U^x \frac{F(n,x)}{\log n} + O(F(2,x)) \qquad \text{(by Abel summation again)}$$

$$= \epsilon \int_2^x \frac{F(u,x)}{\log u} du + o(\int_2^x \frac{F(u,x)}{\log u} du).$$

This holds for all $\epsilon > 0$, so LHS $= o(\int_2^x F(u,x)du/\log u)$.
So LHS of (i) is $\sum_{p \le x} F(p,x) = (1+o(1)) \int_2^x F(u,x)du/\log u$. //

Proof of the Theorem. We prove the case $k = 2$ (Handbuch, 205-8):

$$\pi_2(x) \sim x \log\log x/\log x.$$

The general case follows by a similar but more complicated argument (Handbuch, 208-11), or by induction on k, an argument due to Wright (HW §22.18, Th. 437, 368-71; J, 140-5).

For,

$$\begin{aligned} \pi_2(x) &:= \#\{n \le x : n \text{ has 2 distinct prime factors}\} \\ &= \frac{1}{2}\#\{(p,q) : p,q \text{ distinct primes, } pq \le x\} \end{aligned}$$

($\frac{1}{2}$ because of (p,q) and (q,p)). But $\sum_{p \le x} \pi(x/p)$ is the number of pairs with $p \ne q$, $\pi(\sqrt{x})$ the number of pairs with $p = q$. So by above

$$2\pi(x) = \sum_{p \le x} \pi(x/p) - \pi(\sqrt{x}) = \sum_{p \le x} \pi(x/p) + O(\sqrt{x}/\log x),$$

by PNT or Chebyshev's Upper Estimate. We use the Lemma with

$$F(p,x) := \pi(x/p).$$

For, conditions (i), (ii) are clear. As $\pi(\frac{1}{2}x) \sim \frac{1}{2}x/\log\frac{1}{2}x \sim \frac{1}{2}x/\log x$, (iii) will follow from the relation $(*)$ below:

$$\int_2^x \frac{\pi(x/u)}{\log u} du \sim \frac{2x \log\log x}{\log x}. \qquad (*)$$