m3pm16l28.tex
Lecture 28. 20.3.2012

To prove $(*)$:

$$\int_2^x \frac{\pi(x/u)}{\log u} du = \int_2^{x/2} \frac{\pi(x/u)}{\log u} du$$

(if $u > x/2$, $x/u < 2$, so $\pi(x/u) = 0$)

$$= x \int_2^{x/2} \frac{\pi(v)}{\log x - \log v} \frac{dv}{v^2}$$

($v := x/u$: $2 \leq u = x/v \leq x/2$, so $2 \leq v \leq x/2$). Choose $\epsilon > 0$; for $v \geq V = V(\epsilon)$,

$$|\pi(v) - \frac{v}{\log v}| < \epsilon \frac{v}{\log v}$$

by PNT. So for $x > 2V$,

$$|\int_V^{x/2} \frac{\pi(v)}{\log x - \log v} \frac{dv}{v^2} - \int_V^{x/2} \frac{v/\log v}{\log x - \log v} \frac{dv}{v^2}| < \epsilon \int_V^{x/2} \frac{v/\log v}{\log x - \log v} \frac{dv}{v^2},$$

so $| \int_2^{x/2} ... - \int_2^{x/2} ...| < \epsilon \int_2^{x/2} ... + O(1/\log x)$,
as

$$\int_2^V \frac{v/\log v}{\log x - \log v} \frac{dv}{v^2} = O(1/\log x),$$

etc. Since

$$\int_2^{x/2} \frac{v/\log v}{\log x - \log v} \frac{dv}{v^2} = \int_{\log 2}^{\log x - \log 2} \frac{dw}{w(\log x - w)} \qquad (w := \log v)$$

$$= \frac{1}{\log x} \int_{\log 2}^{\log x - \log 2} \Big(\frac{1}{w} + \frac{1}{\log x - w}\Big) dw \qquad \text{(partial fractions)}$$

$$= \frac{1}{\log x} (\log(\log x - \log 2) - \log\log 2 - \log\log 2 + \log(\log x - \log 2)) \sim \frac{2 \log\log x}{\log x},$$

this gives $(*)$.

By the Lemma, $\pi_2(x) \sim x \log\log x / \log x$, proving the case $k = 2$. //

As each n has at least one prime factor, it is better to work with $k+1$ rather than k. Writing

$$\lambda := \log\log x$$

(so $\lambda \to \infty$ as $x \to \infty$ – though *extremely slowly*):

$$\frac{1}{x}\pi_{k+1}(x) \sim \frac{(\log\log x)^k}{k!\log x} = \frac{e^{-\lambda}\lambda^k}{k!} \quad (k = 0, 1, 2, \ldots) \qquad (\lambda, x \to \infty).$$

Now $\{e^{-\lambda}\lambda^k/k! : k = 0, 1, 2, \ldots\}$ forms the *Poisson distribution* $P(\lambda)$ of Probability Theory, with parameter λ (mean λ, variance λ). So:

Theorem (Landau). The *proportion* of primes $\leq x$ with $k+1$ distinct prime factors is asymptotically Poisson distributed with parameter $\lambda := \log\log x$.

The Poisson distribution is "the signature of randomness", in the *discrete* setting (as here). So this suggests that, *in some sense, the primes are randomly distributed* (hence 'Primes play a game of chance' – see III.10.1 below). This is very surprising: in the ordinary sense, nothing could be less random, or more deterministic, or "God-given", than the primes.

Recall the *prime divisor functions* of II.8: $\omega(n)$ is the number of *distinct* prime divisors of n, $\Omega(n)$ is the number of prime divisors of n counted with multiplicity. It turns out that, as in II.8, ω and Ω behave similarly here. So we may rephrase Landau's theorem above as saying that both proportions $\omega(n)/n$, $\Omega(n)/n$ are asymptotically Poisson distributed with parameter $\lambda := \log\log n$ (recall that $\log\log$ played a key role in II.8 also). Using $X \sim F$ as the usual probabilistic shorthand for "the random variable X has the distribution (function) F", we have

Theorem (Landau's Poisson PNT, 1900).

$$\omega(n)/n \sim P(\log\log n), \qquad \Omega(n)/n \sim P(\log\log n).$$

With some loss of information (the constants C_1, C_2 and the error terms $O(x/\log x)$, we may summarise the Theorem of II.8 for comparison. Using $\sim$ now (with a number after it, not a distribution) to denote "is asymptotic to", one has

Theorem (Hardy and Ramanujan, 1917).

$$\omega(n)/n \sim \log\log n, \qquad \Omega(n) \sim \log\log n.$$