m3pm16l29.tex
Lecture 29. 20.3.2012

10. COMPLEMENTS
1. Probabilistic Number Theory. The situation at the end of III.9 above is sumarised in:
Kac's Dictum (Mark KAC (1914-84)): Primes play a game of chance;
Vaughan's Dictum (R. C. VAUGHAN (1945-)): It's obvious that the primes are randomly distributed – it's just that we don't know what that means yet[1].

A short calculation (involving the probability generating function) shows that if $X_1, \ldots, X_n$ are independent random variables, Poisson $P(1)$ distributed, then their sum is Poisson $P(n)$ (means add; variances add over independent summands; the point is that Poissonianity is preserved). This means that as $n \to \infty$, the *Central Limit Theorem* (CLT) applies: the sum is asymptotically *normally distributed*, or *Gaussian*:

$$P(n) \sim N(n, n)$$

(this statement can be made precise – it is all we need here). Since by Landau's Theorem of III.9 we know that $\omega(n)/n, \Omega(n)/n \sim P(\log \log n)$, this gives

$$\omega(n)/n, \Omega(n)/n \sim N(\log \log n, \log \log n).$$

This is the *Erdös-Kac Central Limit Theorem* of 1939 (Paul ERDÖS (1913-96)).
Note. 1. The Erdös-Kac CLT was completed in 1939, during a seminar given by Kac. Erdös was in the audience, and completed the proof by using *sieve methods*, during the talk.
2. This result marks the "official birth" of the subject of Probabilistic Number Theory, though as III.9 shows the subject really goes back to Landau in 1900 and to Hardy and Ramanujan in 1917. For a textbook account, see e.g. G. TENENBAUM[2]: *Introduction to analytic and probabilistic number theory*, CUP, 1995.

[1] My wife's instant response to this:
(Cecilie) Bingham's Dictum: Primes play a game of chance – we just don't know the rules yet.

[2] Through my 1986 paper with Tenenbaum, I have my Erdös number of 2 (Erdös had Erdös number 0, his collaborators Erdös number 1, etc.)

3. Just as the Landau and Erdös-Kac results correspond to the CLT ("Law of Errors"), the Hardy-Ramanujan result corresponds to the LLN ((Weak) Law of Large Numbers – "Law of Averages"). For background, see e.g. my homepage, link to Stochastic Processes (II.7 L12).

2. Error terms and zero-free regions of ζ.

Error terms in PNT correspond to zero-free regions of ζ (in the critical strip, understood: we knew ζ to be zero-free to the right of the 1-line from II.4, and on it from III.4). Jameson [J] (Ch. 5, Th. 5.1.6, 5.1.8) contains the error terms

$$|\psi(x) - x| \leq Kx \exp\{-c\sqrt{\log x}\}, \qquad |\pi(x) - li(x)| \leq Kx \exp\{-c\sqrt{\log x}\},$$

for some constant $c > 0$, and the zero-free region (Th. 5.3.12)

$$\zeta(s) \neq 0, \qquad s = \sigma + it, \quad |t| \geq 2, \quad \sigma \geq 1 - \frac{1}{840(\log t + 11)}.$$

This material would be included in an 11-week version of this course[3].

Landau (Handbuch, §42) shows that from de la Vallée Poussin's 1896 zero-free region

$$\sigma \geq 1 - \frac{a}{\log t}, \quad t \geq t_0$$

follows

$$\pi(s) - li(x) = O(x \exp\{-\alpha\sqrt{\log x}\}), \qquad \forall \alpha < \sqrt{a}.$$

In the other direction, Pál TURÁN (1910-76) (1950; book of 1984) showed that an error term

$$O(x \exp(-a(\log x)^b))$$

implies a zero-free region

$$\sigma \geq 1 - c(\log(2 + |t|))^{(b-1)/b}.$$

Taking $b = 2/3$, $c = 1/3$ corresponds to the best results known (I. M. VINOGRADOV (1891-1983) in 1958, N. M. KOROBOV in 1958):

$$\psi(x) - x = O(x \exp\{-C(\log x)^{3/5}/(\log\log x)^{1/5}\} \quad (C > 0),$$

$$\sigma \geq 1 - \frac{C}{(\log t)^{2/3}(\log\log t)^{1/3}} \qquad (t \geq t_0).$$

[3]Whether M3PM16/4PM16 is taught for 10 or 11 weeks depends on whether or not Room 340 is needed for examinations in Week 15.