m3pm16l30.tex
Lecture 30. 22.3.2012

3. $\zeta(1+it) \neq 0$.

This is the core of the traditional proofs of PNT. The basic reason is that it is of the form needed to apply *Wiener's general Tauberian theorem*. We have seen the importance of Tauberian theorems in III.6 and III.7 above; Wiener's theorem is the main result in the area. For details, see e.g.,
D. V. WIDDER, *The Laplace transform*, Princeton UP, 1941, Ch. V (esp. §16.5, §16.7).

As above, the bigger the zero-free region, the better the error term (and the smaller, the worse). This suggests that *no* error term – i.e., PNT as we proved it – corresponds to *no* zero-free region (to the left of the 1-line) – i.e., to non-vanishing of ζ *on* the 1-line, which we proved in III.4. This is indeed true: that PNT is equivalent to $\zeta(1+it) \neq 0$ was proved by Ikehara in 1931.

As noted in III.4, the proof we gave is due to Hadamard in 1896 (Handbuch, §45), while a more motivated proof is given by Newman in his book N.

4. Equivalents of PNT.

In Mathematical Logic, all true theorems are equivalent (and all false theorems are equivalent), and Mathematics is a collection of tautologies.

We have a quite different sense of equivalence in mind here. All proofs of PNT are quite hard (elementary proofs are harder than the analytic proofs we gave). In ANT, a statement is called *equivalent to PNT* if either can be easily deduced from the other. For instance, PNT (in the form $\psi(x) \sim x$) implies $M(x) := \sum_{n \leq x} \mu(n) = o(x)$ (Assessed Coursework), and also conversely (Enhanced Coursework). So

$$PNT \Leftrightarrow M(x) = o(x).$$

Similarly (see e.g. [A] Ch. 4, [R] §13.2; cf. III.8), and by above,

$$PNT \Leftrightarrow \sum_1^\infty \mu(n)/n, \qquad PNT \Leftrightarrow \zeta(1+it) \neq 0.$$

5. Primes in arithmetic progressions (APs).

An *arithmetic progression* is a set of the form $nh+k$, $n = 1, 2, \ldots$; w.l.o.g., $(h, k) = 1$.

Theorem (Dirichlet, 1837). There are infinitely many primes in each AP.

Note. 1. This is non-trivial ([A] Ch. 7) – although Euclid's proof that there are infinitely many primes is very easy. Indeed, it is Th. 15* in HW, where the proof is described as being too difficult to be included. But is it a very special case of the result below.
2. To prove this, Dirichlet formulated his *Pigeonhole Principle (Schubfachprinzip)*: if $f : A \to B$ is a map between two finite sets of the same cardinality, then f is surjective (onto) iff f is injective (1-1). This has important applications in Combinatorics (see e.g. Cameron, Ch. 10). It also motivates the Galileo-Dedekind definition of an infinite set: *a set is infinite iff it violates the Pigeonhole Principle.*

Theorem (Dirichlet). For $k > 0$, $(h, k) = 1$ and $x > 1$,

$$\pi(x; h, k) := \sum\nolimits_{p \le x, p \equiv h (mod k)} \sim \frac{li(x)}{\phi(k)}.$$

For proof, see e.g. J Ch. 4, R Ch. 13. This says that, asymptotically, the primes are distributed equally between the residue classes mod k. Error terms are known. All this holds *uniformly* over many APs simultaneously, i.e. in h, k for $k \le (\log x)^u$. See e.g.
T. ESTERMANN, *Introduction to modern prime number theory*, Cambridge Tracts 41, CUP, 1952, §2.1.
6. Elementary proofs of PNT. See III.1 for references. Elementary methods give less good error terms (the best known is $\pi(x) - li(x) = O(xexp\{-(\log x)^{1/6-\epsilon})\})$ (Lavrik and Sobirov, 1973). But by Turán's method (above), this gives a highly non-trivial zero-free region (once thought impossible by elementary methods).
7. Further theory of the Riemann zeta function. See e.g.
E. C. TITCHMARSH, *The theory of the Riemann zeta function*, OUP, 1951 (2nd ed., revised by D. R. HEATH-BROWN, 1986).
Titchmarsh (Ch. II) gives seven methods of proof of the functional equation (III.3). He discusses zero-free regions (CH. III), Ω-theorems – results that show the limits of possible O-theorems (Ch. VIII), and results on the zeros on the critical line (Ch. X). Another valuable source is
A. IVIĆ: *The Riemann zeta function.* Wiley, 1985.

NHB