m3pm16l5.tex
Lecture 5. 24.1.2012

7. THE GAMMA FUNCTION

Recall (M2PM3 II.8, L22, 23) the Euler integral definition:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

The integral converges for $Re\ z > 0$, but from the functional equation $\Gamma(z+1) = z\Gamma(z)$ we can extend Γ successively to $Re\ z > -1, \ldots, Re\ z > -n, \ldots$. This gives the analytic continuation of Γ to the whole complex plane. There, it has poles at $0, \ldots, -n, \ldots$, but no zeros (so $1/\Gamma$ is entire, with zeros at $0, -1, \ldots, -n, \ldots$).

One has the alternative Weierstrass product definition:

$$1/\Gamma(z) = ze^{\gamma z} \prod_{n=1}^{\infty} \{ \Big(1 + \frac{z}{n}\Big) e^{-z/n} \}$$

(M2PM3 Website, link to 'Last year's course', L32, at end). This is the definition preferred in the standard work
[WW] E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed., CUP, 1927/46, Ch. XII.

In WW, Ex. 1 p. 236:

$$\Gamma'(1) = -\gamma$$

(by logarithmic differentiation of the Weierstrass product definition above and putting $z = 1$). Also on WW p.236 (last footnote):

$$\gamma = \int_0^1 (1 - e^{-t}) \frac{dt}{t} - \int_1^\infty \frac{e^{-t}}{t} dt$$

by integration by parts. This also follows from the Euler integral definition by differentiation under the integral sign and putting $z = 1$. Combining:

$$\gamma = -\Gamma'(1) = -\int_0^\infty e^{-x} \log x dx$$

(HW, (22.8.2), p.351). We shall use this in II.7 (as do Hardy and Wright) in the proof of Mertens' Theorem.

§8. EULER'S SUMMATION FORMULA

This relates to the close connection between sums and integrals. We give only what is needed later (III.3: analytic continuation of ζ). This is a special case of the Euler-Maclaurin sum(mation) formula (see e.g. WW §7.21).

Theorem (i). For m, n integers, f differentiable on $[m, n]$,

$$\sum_{m+1}^{n} f(r) - \int_m^n f = \int_m^n (t - [t]) f'(t) dt.$$

Proof. $[.] = r - 1$ on $[r - 1, r)$. Integrating by parts,

$$\int_{r-1}^{r} (t - r + 1) f'(t) dt = [(t - r + 1) f(t)]_{r-1}^{r} - \int_{r-1}^{r} f = f(r) - \int_{r-1}^{r} f.$$

Sum over $r = m + 1$ to n. //

Th. (ii). In Th. (i),

$$\frac{1}{2} f(m) + \sum_{m+1}^{n-1} f(r) + \frac{1}{2} f(n) - \int_m^n f = \int_m^n (t - [t] - \frac{1}{2}) f'(t) dt.$$

Proof. As above, or from (i). //

Th. (iii). If m is an integer, x real, f differentiable on $[m, x]$,

$$\sum_{m<r\leq x} f(r) - \int_m^x f = aint_m^x (t - [t]) f'(t) dt - (x - [x]) f(x).$$

Proof. Let $n := [x]$. In (i), add

$$\int_n^x (t - n) f'(t) dt = [(t - n) f(t)]_n^x - \int_n^x f = (x - n) f(x) - \int_n^x f.$$

Cor. If f is differentiable on $[1, \infty)$ and $\sum_1^\infty f(r), \int_1^\infty f(t) dt$ both converge,

$$\sum_1^\infty f(r) - \int_1^\infty f(t) dt = f(1) + \int_1^\infty (t - [t]) f'(t) dt = \frac{1}{2} f(1) + \int_1^\infty (t - [t] - \frac{1}{2}) f'(t) dt.$$

Proof. Take $m = 1$ and let $n \to \infty$. //

As in the Integral Test (I.4), if $f\downarrow$ the difference $S(x)-I(x)$ of the sume- and integral converges even if both diverge.

Th. If $f(x)\downarrow 0$ as $x\to\infty$, $S(x) := \sum_{m<r\le x} f(r)$, $I(x) := \int_m^x f(t)dt$, then
(i) $S(x)-I(x)\to L$ as $x\to\infty$, where $L := f(1)+\int_1^\infty (t-[t])f'(t)dt$;
(ii) $0\le L\le f(1)$;
(iii) For $x\ge 1$, $S(x) = I(x)+L+q(x)$, $|q(x)|\le f(x)$. For $x=n$ integer, $0\le q(x)\le f(x)$.

Proof. Take $m=1$ in Th. (ii) above:

$$S(x)-I(x) = f(1)+\int_1^x (t-[t])f'(t)dt-(x-[x])f(x).$$

As $f\downarrow 0$, $\int_x^\infty f'(t)dt = [f]_t^\infty = -f(x)$. As $0\le t-[t]<1$, $\int_1^\infty (t-[t])f'(t)dt$ converges, with value in $[-f(1),0]$. So

$$S(x)-I(x)\to L\in[0,f(1)].$$

And $S(x)-I(x) = L-\int_x^\infty (t-[t])f'(t)dt-(x-[x])f(x) = L+J(x)-F(x)$, say, where as $f\downarrow$

$$0\le J(x)\le -\int_x^\infty f' = f(x).$$

Also $0\le F(x)\le f(x)$, so $|J(x)-F(x)|\le f(x)$ (and $F(n)=0$). //

Cor. (J Prop.1.4.11 p.25, A p.56).

$$\sum_1^n 1/r-\log n\to\gamma = 1-\int_1^\infty \frac{t-[t]}{t^2}dt \qquad (n\to\infty),$$

$$0<\gamma<1, \qquad \sum_{1\le r\le x} 1/r = \log x+\gamma+q(x), \qquad |q(x)|\le 1/x.$$