m3pm16l6.tex
**Lecture 6. 26.1.2012**

## II. Arithmetic Functions and Dirichlet Series

### §1. Dirichlet Series

*Defn.* An *arithmetic function* $a \mapsto a_n$ or $a(n)$ is a map from $\mathbf{N}$ to $\mathbf{R}$ or $\mathbf{C}$.
*Notation:* For $s \in \mathbf{C}$ we write $s = \sigma + it$.
The *Dirichlet series* of $a$ is the function $\sum_{n=1}^{\infty} a_n/n^s$.

While the region of convergence of a power series is a *disc* where it is also *absolutely convergent*, the regions of convergence and absolute convergence of a Dirichlet series are *half-planes*, possibly different.

**Theorem (Half Plane of Absolute Convergence).**
(i) If $\sum_1^{\infty} a_n/n^s$ is absolutely convergent for $s = \alpha$, real, it is also convergent for $s = \sigma + it, \sigma \geq \alpha$.
(ii) There exists $\sigma_a$, the *abscissa of absolute convergence*, such that $\sum_1^{\infty} a_n/n^s$ is absolutely convergent for $\sigma > \sigma_a$, and not absolutely convergent for $\sigma < \sigma_a$.

*Proof.* (i) $n^s = n^{\sigma+it} = n^{\sigma} e^{it \log n}$, so $|n^s| = n^{\sigma}$. So for $\sigma \geq \alpha, |a_n/n^s| = |a_n|/n^{\sigma} \leq |a_n|/n^{\alpha}$, and we know this converges absolutely.
(ii) Let

$$E := \{\alpha \in \mathbf{R} : \sum |a_n|/n^{\alpha} < \infty\}, \qquad \sigma_a = \inf\{E\}.$$

In (i), given $\alpha \in E$, so $E \neq \phi$. If $\sigma > \sigma_a$, $\exists \alpha \in E$ with $\alpha < \sigma$, and then by (i), $\sigma \in E$, so $\sum a_n/n^{\sigma}$ is absolutely convergent. Clearly, if $\sigma < \sigma_a$, then $\sigma \notin E$, as $\sigma_a$ is an infimum of the set. (Observe that $\sigma_a$ is a Dedekind cut.) //

### Abel Summation Formula for Dirichlet Series

Again, $A(x) := \sum_{n \leq x} a_n$. Abel's summation formula for $f(x) = 1/x^s, f'(x) = -s/x^{1+s}$ gives

$$\sum_{n \leq x} a_n/n^s = \frac{A(x)}{x^s} + s \int_1^x \frac{A(x)}{x^{1+s}} dx. \qquad (*)$$

So if $s \neq 0$ and $A(n)/x^s \to 0$ at $\infty$, if one of $\sum_1^{\infty} a_n/n^s$ and $s \int_1^{\infty} A(x)/x^{1+s} dx$ converges, both do to the same value (by the Integral Test). Similarly,

$$\sum_{n > x} \frac{a_n}{n^s} = -\frac{A(x)}{x^s} + s \int_x^{\infty} \frac{A(x)}{x^{1+s}} dx. \qquad (**)$$

We call $\int_1^\infty f(x)/x^{1+s}dx$ a *Dirichlet integral* (essentially equivalent to Dirichlet series). We shall only deal with Dirichlet integrals for functions that are constant except for jumps at the integers, where they have left and right limits. This ensures Riemann integrability.

**Proposition**. If $A(x) := \sum_{n \le x} a_n$ has $|A(x)| \le Mx^\alpha (n \ge 1, \alpha \ge 0)$, the Dirichlet series $F(s) := \sum_{n=1}^\infty a_n/n^s$ converges for $s = \sigma + it, \sigma > \alpha$. Write $F_x(s) := \sum_{n \le x} a_n/n^s$. Then

$$|F(s)| \le \frac{M|s|}{\sigma - \alpha}; \qquad |F(s) - F_x(s)| \le \frac{M}{x^{\sigma-\alpha}} \left( \frac{|s|}{\sigma - \alpha} + 1 \right).$$

*Proof.* On the RHS of $(*)$, $|A(x)/x^s| \le M/x^{\sigma-\alpha}$. Then

$$|s| \int_1^x \frac{A(x)}{x^{1+s}} dx \le |s| \int_1^\infty \frac{M}{x^{\sigma-\alpha+1}} dx = \frac{M|s|}{\sigma - \alpha} \left( 1 - \frac{1}{x^{\sigma-\alpha}} \right) \le \frac{M|s|}{\sigma - \alpha}.$$

Letting $x \to \infty$ in $(*)$ gives $|F(s)| \le M|s|/(\sigma - \alpha)$. Similarly for $(**)$. //

**Theorem (Half-plane of convergence)**.
(i) If $\sum_1^\infty a_n/n^\alpha$ converges for some real $\alpha$, the series $\sum_1^\infty a_n/n^s$ converges for $s = \sigma + it$, $\sigma > \alpha$.
(ii) Consequently, there exists $\sigma_c$, the *abscissa of convergence* (possibly $\pm\infty$) such that $\sum_1^\infty a_n/n^s$ converges for $\sigma > \sigma_c$ and diverges for $\sigma < \sigma_c$.
(iii) $\sigma_c \le \sigma_a \le \sigma_c + 1$.

*Proof.*(i) Write $b_n := a_n/n^\alpha$, $B(x) := \sum_{n \le x} b_n$. Then $\sum b_n$ converges, so is bounded: say $|B(x)| \le M$. Take $\alpha = 0$ in the Prop. above: $\sum b_n/n^s$ converges ($Res > 0$). So $\sum a_n/n^s = \sum b_n/n^{s-\alpha}$ converges ($\sigma > \alpha$).
(ii) This follows as with $\sigma_a$ above.
(iii) $\sigma_c \le \sigma_a$ as absolute convergence implies convergence (so the half-plane of absolute convergence $\subset$ the half-plane of convergence).

$$|a_n/n^s| = |b_n/n^{s-\alpha}| \le M/n^{\sigma-\alpha}.$$

So for $\sigma > \alpha + 1$, $\sum a_n/n^s$ is absolutely convergent by the Comparison Test ($\sum 1/n^c$ converges for $c > 1$). So $\sigma_a \le \alpha + 1$.
This holds for every $\alpha > \sigma_c$. So $\sigma_a \le \sigma_c + 1$. //