m3pm16prob5.tex

## M3PM16/M4PM16 PROBLEMS 5. 16.2.2012

Q1. By considering the series expansion of $-\log(1-x)$, or otherwise, show that $\prod(1-1/p)$ diverges.

Q2. Use the divergence of $\prod(1-1/p)$ to show (by considering the number $N(x,r)$ of $n \leq x$ not divisible by any of the first $r$ primes $p_k$, or otherwise) that

$$\pi(x) = o(x).$$

(This bound is much weaker than PNT $\pi(x) \sim li(x) \sim x/\log x$, but is useful and non-trivial.)

Q3. Show that if $c := a * b$ and $b$ are multiplicative, then $a$ is multiplicative.

Q4. Define the *Bernoulli numbers* $B_n$ by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}$$

(this is the *generating function (GF)* of the $B_n$; GFs are a common way to define a series). Show that
(i) $B_0 = 1$, $(n+1)B_n = -\sum_{k=0}^{n-1} \binom{n+1}{k} B_k$ $(n \geq 1)$; $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, $B_6 = 1/42$.
(ii)

$$z \cot z = 1 + \sum_{n=2}^{\infty} B_n (2iz)^n / n!,$$

and hence $B_n = 0$ for all odd $n > 1$.
(iii) For $n$ a positive integer,

$$\zeta(2n) = (-)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}.$$

NHB