m3pm16soln3.tex

M3PM16/M4PM16 SOLUTIONS 3. 9.2.2012

Q1 (J p.6 Ex. 3). (i) Assume inductively that $p_k \le 2^{2^{k-1}}$ $(k = 1, \ldots, n)$ (the induction starts, as $p_1 = 2$). Then

$$p_1 \ldots p_n \le 2^{1+2+\ldots+2^{n-1}} = 2^{2^n - 1}.$$

Following Euclid's proof, consider $N := p_1 \ldots p_n + 1$.
This is not divisible by any of $p_1, \ldots, p_n$. So if p is any prime in its prime-power factorization (FTA), $p|N$, so $p \le N$. Also $p_{n+1} \le p$ (as by above p is not one of the first n primes). So

$$p_{n+1} \le p \le N = p_1 \ldots p_n + 1 \le 2^{2^n - 1} + 1 \le 2^{2^n}.$$

This completes the induction, proving $p_n \le 2^{2^{n-1}}$ for all n.
(ii) Given x, let n be the integer with $2^{2^{n-1}} \le x < 2^n$. Then

$$\pi(2^{2^{n-1}}) \le \pi(x) < \pi(2^{2^n}).$$

$$\pi(x) := \sum_{p \le x} 1 = \sum_{k: p_k \le x} 1 \ge \sum_{k: 2^{2^{k-1}} \le x} 1,$$

by (i). But $2^{2^{k-1}} \le x$ iff $2^{k-1} \log 2 \le \log x$, $2^{k-1} \le \log x / \log 2$,
iff $(k-1) \log 2 \le \log\log x - \log\log 2$, iff $k \le (\log\log x)/(\log 2) - (\log\log 2)/(\log 2) + 1$. So

$$\pi(x) \ge 1 - \frac{\log\log 2}{\log 2} + \frac{\log\log x}{\log 2} > \frac{\log\log x}{\log 2},$$

as $1 - (\log\log 2)/(\log 2) > 1 > 0$ (see HW Th. 10 p.12 for the slightly less precise $\pi(x) \ge \log\log x$).

Q2 (HW §2.6, Th. 20, p.16-7). (i) If $2, 3, \ldots, p_j$ are the first j primes and N is the number of $n \le x$ not divisible by any $p > p_j$: each such n is of the form

$$n = n_1^2 m, \qquad m = p_1^{c_1} \ldots p_j^{c_j}, \qquad c_i = 0 \text{ or } 1$$

(any even powers of p_i being absorbed in n_1^2). There are 2^j choices of the powers c_i, so $\#m = 2^j$. Also $n_1 \le \sqrt{n} \le \sqrt{x}$, so $\#n_1 \le \sqrt{x}$. Combining,

$$N(x) = \#n \le \#m.\#n_1 = 2^j \sqrt{x} : \qquad N(x) \le 2^j \sqrt{x}.$$

(ii) If $\sum 1/p < \infty$: choose j so large that $\sum_{j+1}^{\infty} 1/p_k < 1/2$. The number of $n \leq x$ divisible by p is $[x/p] \leq x/p$. So the number of $n \leq x$ divisible by at least one of the p_k $(k \geq j+1)$ is $\leq x \sum_{j+1}^{\infty} 1/p_k < x/2$. Combining this with (i):

$$\frac{1}{2}x < N(x) \leq 2^j \sqrt{x} : \qquad \sqrt{x} \leq 2^{j+1} : \qquad x \leq 2^{2j+2}.$$

This is false for large enough x (j is fixed). This contradiction gives $\sum 1/p$ diverges.

(iii) Take $j = \pi(x)$. So $p_{j+1} > x$, and $N(x) = x$. Then (i) gives

$$x = N(x) \leq 2^{\pi(x)} \sqrt{x} : \qquad 2^{\pi(x)} \geq \sqrt{x}.$$

Take logs:

$$\pi(x) \geq \frac{\log x}{2 \log 2}.$$

(iv) Taking $x = p_n$: $\pi(x) = n$: $2^n \geq \sqrt{p_n}$, so $p_n \leq 4^n$.

Q3 (HW Th. 5 p.5). If $2, 3, \ldots, p$ are all the primes up to p, then all numbers n up to p are divisible by at least one of these primes, p' say, by FTA:

$$p' | n, \qquad n = p' r',$$

say. So if $q := 2.3.5 \ldots p = \prod p'$, then all the $p - 1$ numbers $q + 2, q + 3, q + 4, \ldots, q + p$ are composite: for each is of the form

$$q + n = q + p' r' = p'. \prod p'' + p' r' = p'(r' + \prod p''),$$

where the product is over the primes other than p in the prime-power factorisation of n (counted with multiplicity), and any repetitions of p. So this string of $p - 1$ consecutive numbers forms (part of) a gap between primes. There are arbitrarily large p (Euclid), so arbitrarily long gaps between primes.

Q4 (HW, Th. 11 p.13). Again as in Euclid, write $q := 2^2.3, \ldots p - 1$. Then $4 | q$, so $q = 4n + 3$ for some n (residue $3 = -1 \bmod 4$), and q is not divisible by any of the primes up to p. It cannot be a product of primes of the form $4n + 1$, or it too would be of this form. So q is of the form $4n + 3$, and there are infinitely may such q, one for each p.

Note. There are also infinitely many primes of the form $4n + 1$, but this is harder (HW, p.13). More is true: from Dirichlet's PNT for primes in AP (III.10.7), 'half the primes are $4n + 1$, half are $4n + 3$'.

NHB