m3pm16soln5.tex

M3PM16/M4PM16 SOLUTIONS 5. 23.2.2012

Q1. $-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots$. So

$$0 < -\log(1-1/p) = \frac{1}{2p^2} + \frac{1}{3p^3} + \ldots < \frac{1}{2p^2} + \frac{1}{2p^3} + \ldots = \frac{1}{2p(p-1)},$$

summing the GP. Also

$$\sum_p \frac{1}{p(p-1)} < \sum_n \frac{1}{n(n-1)} < \infty.$$

So by the Comparison Text,

$$\sum_p \{-\log(1-1/p) - 1/p\} \text{ converges.}$$

But (Euler, II.4) $\sum 1/p$ diverges. So $\sum\{-\log(1-1/p)\}$ diverges also. That is, the infinite product $\prod(1-1/p)$ diverges to 0 (I.5).

Q2 (HW, 4th ed., §22.7 – I find this proof more transparent than the one in the 5th ed.). With $N(x,r)$ the number of $n \le x$ not divisible by any of the first r primes p_k, then

$$\pi(x) \le N(x,r) + r$$

(a prime $p \le x$ is either one of the first r or not divisible by any of the first r). By Inclusion-Exclusion (Problems 4 Q2),

$$N(x,r) = [x] - \sum_i [x/p_i] + \sum_{ij} [x/p_i p_j] \ldots$$

The number of square brackets is

$$1 + \binom{r}{1} + \binom{r}{2} + \ldots = (1+1)^r = 2^r.$$

Replacing each [.] by . introduces an error of < 1, so

$$N(x,r) < x - \sum_i x/p_i + \sum_{ij} x/p_i p_j \ldots + 2^r = x \prod_1^r (1 - 1/p_k) + 2^r.$$

Combining,

$$\pi(x) \le x \prod_1^r (1 - 1/p_k) + 2^r + r : \qquad \pi(x)/x \le \prod_1^r (1 - 1/p_k) + (2^r + r)/x.$$

As the product diverges (Q1), $\prod_1^r$ can be made arbitrarily small by taking r large enough. Then letting $x \to \infty$ gives $\pi(x)/x \to 0$. //

Q3 (A, Th. 2.15 p.35-6). By contradiction: we assume a is not multiplicative and deduce that $a * b$ is not multiplicative. Let $c := a * b$. As a is not multiplicative, there are positive integers m, n with $(m, n) = 1$ but $a(mn) \neq a(m)a(n)$. Choose the pair m and n with mn as small as possible.

If $mn = 1$, then $a(1) \neq a(1)a(1)$, so $a(1) \neq 1$. As $b(1) = 1$ (b multiplicative) and $c(1) = a(1)b(1)$ ($c := a * b$ is multiplicative), $c(1) = a(1)b(1) = a(1) \neq 1$, this shows that $c = a * b$ is not multiplicative, a contradiction.

If $mn > 1$, then by minimality of mn, $a(m'n') = a(m')a(n')$ for all coprime m', n' with $m'n' > mn$. So (as in II.3 Prop.)

$$\begin{aligned} c(mn) &= \sum_{j|m,k|n,jk<mn} a(jk)b(mn/jk) + a(mn)b(1) \\ &= \sum_{j|m,k|n,jk<mn} a(j)a(k)b(m/j)b(n/k) + a(mn) \\ &= \sum_{j|m} a(j)b(m/j) \sum_{k|n} a(k)b(n/k) - a(m)b(n) + a(mn) \\ &= c(m)c(n) - a(m) + a(mn). \end{aligned}$$

As $a(mn) \neq a(m)a(n)$, this gives $c(mn) \neq c(m)c(n)$, contradicting multiplicativity of c. //

Q4. (i) (see e.g. R, 229-231). Multiplying by $e^z - 1$: $z = (\sum_{i=0}^{\infty} B_i z^i/i!)(\sum_{j=1}^{\infty} z^j/j!)$. Equate coefficients: the z term gives $B_0 = 1$. For higher terms,

$$0 = \sum_{i+j=n,j>0} \frac{B_i}{i!}.\frac{1}{j!} = \sum_{i=0}^{n-1} \frac{B_i}{i!(n-i)!} = \frac{1}{n!} \sum_{i=0}^{n-1} B_i \binom{n}{i} : \qquad 0 = \sum_{i=0}^{n-1} B_i \binom{n}{i}.$$

Replacing n by $n + 1$ and picking out the leading term,

$$(n + 1)B_n = - \sum_{i=0}^{n-1} B_i \binom{n+1}{i},$$

as required. The values $B_1, \ldots, B_6$ can be checked from this.
(ii) From Euler's formulae $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$, $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$,

$$x \cot x = ix \frac{(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix})} = ix \frac{(e^{2ix} + 1)}{e^{2ix} - 1)} = ix\Big(1 + \frac{2}{e^{2ix} - 1)}\Big).$$

As $B_0 = 1$ and $B_1 = -\frac{1}{2}$, this gives

$$x \cot x = 1 + \sum_2^\infty B_n (2ix)^n / n!$$

As the LHS is even, so is the RHS, so $B_n = 0$ for all *odd* $n > 1$. So

$$x \cot x = 1 + \sum_1^\infty B_{2n} (-)^n (2x)^{2n} / (2n)! \qquad (a)$$

(iii) From M2PM3 (Lecture 32, formula (ii), at the end),

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^\infty \frac{1}{z^2 - n^2\pi^2}.$$

So

$$z \cot z = 1 - 2z^2 \sum_{n=1}^\infty \frac{1}{n^2\pi^2} = 1 - 2 \sum_{n=1}^\infty (z/n\pi)^2 / (1 - (z/n\pi)^2) = 1 - 2 \sum_{n=1}^\infty \sum_{m=1}^\infty (z/n\pi)^{2m},$$

expanding as a geometric series. Inverting the order of summation (as we may by absolute convergence for $|z| < 1$, and can then use analytic continuation),

$$z \cot z = 1 - 2 \sum_{m=1}^\infty \sum_{n=1}^\infty (z/n\pi)^{2m} = 1 - 2 \sum_{m=1}^\infty (z/\pi)^{2m} \zeta(2m). \qquad (b)$$

Equating coefficients in (a), (b),

$$2(2n)!\zeta(2n) = (-)^{n+1} (2\pi)^{2n} B_{2n}.$$

This recovers $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, both proved in M2PM3, and also gives the next such result, $\zeta(6) = \pi^6/945$. NHB