m3pm16soln6.tex

M3PM16/M4PM16 SOLUTIONS 6. 30.2.2012

Q1 (J, Ex.1 p.37). By Abel summation, with $a(n) := I_P(n)$ (I for indicator function, P for the set of primes), so $A(x) = \sum_{p \le x} 1 = \pi(x)$,

$$f(x) = \frac{1}{x \log x}, \quad f'(x) = -\frac{1}{x^2 \log x} + \frac{1}{x}.-\frac{1}{\log^2 x}.\frac{1}{x}, \quad -f'(x) = \frac{1}{x^2 \log x} + \frac{1}{x^2 \log^2 x},$$

$\sum_{n \le x} a(n) f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt$ gives

$$\sum_{p \le x} \frac{1}{p \log p} = \frac{\pi(x)}{x \log x} + \int_2^x \frac{\pi(t)}{t^2 \log t} dt + \int_2^x \frac{\pi(t)}{t^2 \log^2 t} dt$$

($\pi(x) = 0$ for $x < 2$ as 2 is the smallest prime). The first term is $o(1/\log x) = o(1)$ (this only needs $\pi(x) := \sum_{p \le x} 1 \le x$); the third term is negligible w.r.t. the second, which by Chebyshev's Upper Estimate is of order

$$\int_2^x \frac{(t/\log t)}{t^2 \log t} dt = \int_2^x \frac{dt}{t \log^2 t} = \int_{\log 2}^{\log x} du/u^2 \le \int_c^\infty du/u^2 < \infty.$$

So $\sum 1/(p \log p)$ converges.

Q2. (i)

$$\log \sin z = \log z + \sum_1^\infty \log(1 - \frac{z^2}{n^2 \pi^2}),$$

$$\cot z = 1/z - \sum_1^\infty \frac{\frac{2z}{n^2 \pi^2}}{1 - \frac{z^2}{n^2 \pi^2}}.$$

Multiplying by z and expanding the geometric series,

$$z \cot z = 1 - 2 \sum_{n=1}^\infty \sum_{k=1}^\infty (z/n\pi)^{2k}. \qquad (1)$$

As $\sum_1^\infty 1/n^{2k} = \zeta(2k)$,

$$z \cot z = 1 - 2 \sum_{k=1}^\infty z^{2k} \zeta(2k)/\pi^{2k}.$$

(ii)

$$\cot z = \cos z / \sin z = \frac{1}{2}(e^{iz} - e^{-iz}) / \frac{1}{2i}(e^{iz} - e^{-iz}) = i\frac{e^{2iz}+1}{e^{2iz}-1} = i + \frac{2i}{e^{2iz}-1}.$$

So

$$z \cot z = iz + \frac{2iz}{e^{2iz}-1} = iz + 1 - iz + \sum_{2}^{\infty} (2iz)^n B_n / n! = 1 + \sum_{1}^{\infty} B_{2k} (-)^k \frac{2^{2k} z^{2k}}{(2k)!}. \quad (2)$$

Equating coefficients of z^{2k} in (1), (2): For $k = 1, 2, \ldots$,

$$-2\zeta(2k)/(\pi^{2k} = (-)^k B_{2k} 2^{2k}/(2k)! : \qquad \zeta(2k) = (-)^{k+1} (2\pi)^{2k} B_{2k} / (2(2k)!).$$

NHB