m3pm16comp.tex

Handout: Further Complex Analysis.

These results will be needed for the proof of PNT with remainder term.

The Gamma function.

We return to the Gamma function of I.7.

Stirling's formula. Recall that for $n \in \mathbb{N}$ $\Gamma(n+1) = n!$ – the Gamma function is a continuous extension of the factorial. Then (James STIRLING (1692-1770) in 1730)

$$n! \sim \sqrt{2\pi} e^{-n} n^{n+\frac{1}{2}} \qquad (n \to \infty).$$

In terms of the Gamma function,

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}} \qquad (x \to \infty).$$

We shall need an estimate for $\Gamma(z)$ with z complex. Recall that Γ has poles at $0, -1, -2, \ldots$ but no zeros, so $1/\Gamma$ is entire (with zeros at $0, -1, -2, \ldots$). For $\delta > 0$, write $D_\delta := \{z \in \mathbb{C} : -\pi + \delta < arg z < \pi - \delta,\ |z| > 1\}$ (so we can 'go off to infinity' avoiding the poles on the negative real axis). Then

$$\Gamma(z) \sim \sqrt{2\pi} e^{-z} z^{z-\frac{1}{2}} \Big(1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots\Big) \qquad (z \in D_\delta,\ |z| \to \infty)$$

(the RHS is an *asymptotic expansion*). This yields an asymptotic expansion for $\log \Gamma(z)$ (involving the Bernoulli numbers – see e.g. WW, 12.33), and hence (all we shall need)

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O_\delta(1/|z|) \qquad (z \in D_\delta). \qquad (St)$$

It can be shown that the error term here has derivative $O_\delta(1/|z|^2)$ (as one would expect). So differentiating, the error term is negligible, and one obtains

$$\Gamma'(z)/\Gamma(z) = \log z + O_\delta(1/|z|) \qquad (z \in D_\delta).$$

This logarithm occurs again in the zero-free region for $\zeta(s)$ that we obtain, and this in turn gives us our error term in PNT.

We can also estimate Γ in vertical strips. For this, only the leading term

$z^{z-\frac{1}{2}} = \exp\{(z-\frac{1}{2})\log z\}$ in Stirling's formula matters, and only large t matters. One obtains:

$$|\Gamma(\sigma+it)| << |t|^{\beta-\frac{1}{2}}e^{-\frac{1}{2}\pi t} \qquad (\alpha \le \sigma \le \beta,\ t>1),$$

where the constant implied in the $<<$ depends on α, β. For here,
$|(z-\frac{1}{2})\log z| = (\sigma-\frac{1}{2})\log r - \theta t$; as $t \to \infty$, $r \sim t$, $\theta \uparrow \frac{1}{2}\pi$, so this is $<< \log(t^{\beta-\frac{1}{2}}.e^{-\frac{1}{2}\pi t})$.

Entire functions of order 1.

Hadamard, in the course of his proof of PNT using Complex Analysis in 1896, developed a theory of factorization of entire functions. This is standard Complex Analysis (see e.g. Ahlfors [Ahl], 5.3.2) rather than Number Theory, so we shall quote what we need. The *order* of an entire function f is the least a for which

$$|f(z)| = O_\delta(\exp\{|z|^{a+\delta}\}) \qquad (|z| \to \infty).$$

We shall only need the case of *order 1*, and that only for Γ and ζ. Hadamard's factorization theorem for entire functions f of order 1 states that
(i) f can be written as

$$f(z) = z^r e^{Az+B} \prod_{\rho \neq 0} \{(1-z/\rho)e^{z/\rho}\},$$

where r is the order of the zero at 0 (if any), A, B are constants, and ρ runs through the other zeros (if any);
(ii)

$$\sum_{\rho \neq 0} |\rho|^{-1-\delta}$$

converges for any $\delta > 0$, and for any $X > 1$
(iii)

$$\sum_{|\rho| \ge X} <<_\delta X^{-\frac{1}{2}\delta}.$$

Taking $\delta = 1$ in (ii) gives $\sum |\rho|^{-2}$ converges, whence the product in (i) converges. The proof involves Jensen's formula from Complex Analysis.

We have already met one instance of this, in Weierstrass's product definition of Γ (I.7). In lectures (III.9), we apply it to ζ.