M3PM16/M4PM16 SOLUTIONS TO ASSESSED COURSEWORK, 2013

MERTENS' THEOREM WITH REMAINDER.

$$\prod_{p\le x}(1-\frac{1}{p}) = \frac{e^{-\gamma}}{\log x}\Big(1+O(\frac{1}{\log x})\Big).$$

Proof. As in L14: write

$$\Sigma := \sum_p\Big(\log(1-\frac{1}{p})+\frac{1}{p}\Big).$$

From Mertens' Second Theorem and the Constants Lemma (as in L14),

$$\sum_{p\le x}\frac{1}{p} = \log\log x + C_1 + O(1/\log x) = \log\log x + \gamma + \Sigma + O(1/\log x).$$

Now

$$\sum_{p\le x}\log(1-\frac{1}{p})^{-1} = \sum_{p\le x}\frac{1}{p} + \sum_{p\le x}\Big(\log(1-\frac{1}{p})^{-1}-\frac{1}{p}\Big).$$

From the power-series expansion for $\log(1-x)$, the second sum on the right is

$$-\Sigma - \sum_{p>x}\ldots = (\sum_p - \sum_{p>x})\ldots = \sum_p\sum_{k=2}^{\infty}\frac{1}{kp^k} + O(\sum_{p>x}p^{-2}).$$

The error term is

$$<< \sum_{p>x}p^{-2} < \sum_{n>x}n^{-2} << 1/x.$$

So the second sum is $-\Sigma + O(1/x)$. So by Mertens' Second Theorem,

$$\log\prod_{p\le x}(1-\frac{1}{p})^{-1} = \sum_{p\le x}\log(1-\frac{1}{p})^{-1} = \log\log x + (\Sigma+\gamma) + O(1/\log x) - \Sigma + O(1/x)$$

$$= \log\log x + \gamma + O(1/\log x).$$

Since $e^z = 1 + O(|z|)$ as $z\to 0$, exponentiating gives

$$\prod_{p\le x}(1-\frac{1}{p})^{-1} = \exp\{\log\log x + \gamma + O(1/\log x)\}$$

$$= e^{\gamma}.\log x.(1+O(1/\log x)) = e^{\gamma}\log x + O(1):$$

$$\prod_{p\le x}(1-\frac{1}{p}) = \frac{e^{-\gamma}}{\log x}.(1+O(1/\log x)).$$

NHB