m3pm16l5.tex

**Handout (I.8): Euler Summation and the Integral Test**

As in the Integral Test (I.4), if $f \downarrow$ the difference $S(x) - I(x)$ of the sum and integral converges even if both diverge.

**Th.** If $f(x) \downarrow 0$ as $x \to \infty$, $S(x) := \sum_{m<r\leq x} f(r)$, $I(x) := \int_m^x f(t)dt$, then
(i) $S(x) - I(x) \to L$ as $x \to \infty$, where $L := f(1) + \int_1^\infty (t - [t])f'(t)dt$;
(ii) $0 \leq L \leq f(1)$;
(iii) For $x \geq 1$, $S(x) = I(x) + L + q(x)$, $|q(x)| \leq f(x)$. For $x = n$ integer, $0 \leq q(x) \leq f(x)$.

*Proof.* Take $m = 1$ in Th. (ii) above:

$$S(x) - I(x) = f(1) + \int_1^x (t - [t])f'(t)dt - (x - [x])f(x).$$

As $f \downarrow 0$, $\int_x^\infty f'(t)dt = [f]_t^\infty = -f(x)$. As $0 \leq t - [t] < 1$, $\int_1^\infty (t - [t])f'(t)dt$ converges, with value in $[-f(1), 0]$. So

$$S(x) - I(x) \to L \in [0, f(1)].$$

And $S(x) - I(x) = L - \int_x^\infty (t - [t])f'(t)dt - (x - [x])f(x) = L + J(x) - F(x)$, say, where as $f \downarrow$

$$0 \leq J(x) \leq -\int_x^\infty f' = f(x).$$

Also $0 \leq F(x) \leq f(x)$, so $|J(x) - F(x)| \leq f(x)$ (and $F(n) = 0$). //

**Cor.** (J Prop.1.4.11 p.25, A p.56).

$$\sum_1^n 1/r - \log n \to \gamma = 1 - \int_1^\infty \frac{t - [t]}{t^2} dt \qquad (n \to \infty),$$

$$0 < \gamma < 1, \qquad \sum_{1\leq r\leq x} 1/r = \log x + \gamma + q(x), \qquad |q(x)| \leq 1/x.$$