M3PM16/M4PM16 SOLUTIONS TO EXAMINATION 2012

Q1. (i)

$$T_x := \prod_{p\le x} 1/(1-\frac{1}{p}) = \sum\nolimits_x^* 1/n \ge \sum\nolimits_1^x 1/n \ge \log x,$$

where $\sum_x^*$ denotes a sum over all n with all prime factors $\le x$. But for $0 < x < 1$

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots < x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \ldots = x + \frac{\frac{1}{2}x^2}{1-x},$$

so for $y > 1$

$$-\log(1-1/y) - 1/y < \frac{1}{2y^2(1-1/y)} = \frac{1}{2y(y-1)}.$$

So if $S_x := \sum_{p\le x} 1/p$,

$$\log T_x - S_x = \sum_{p\le x}\Big(-\log(1-\frac{1}{p}) - \frac{1}{p}\Big) < \frac{1}{2}\sum_{p\le x} 1/(p(p-1)) < \frac{1}{2}\sum_{n=2}^{\infty} 1/(n(n-1)) = \frac{1}{2}$$

(by partial fractions – the sum telescopes). So

$$S_x \ge \log T_x - \tfrac{1}{2} \ge \log\log x - \tfrac{1}{2}. \qquad \textbf{[8]}$$

Letting $x \to \infty$: $\sum_p 1/p$ diverges. **[2]**

(ii) By Abel summation, with $a(n) := I_P(n)$ (I for indicator function, P for the set of primes), so $A(x) = \sum_{p\le x} 1 = \pi(x)$,

$$f(x) = \frac{1}{x\log x}, \quad f'(x) = -\frac{1}{x^2\log x} + \frac{1}{x}.-\frac{1}{\log^2 x}.\frac{1}{x}, \quad -f'(x) = \frac{1}{x^2\log x} + \frac{1}{x^2\log^2 x},$$

$\sum_{n\le x} a(n)f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt$ gives

$$\sum_{p\le x}\frac{1}{p\log p} = \frac{\pi(x)}{x\log x} + \int_2^x \frac{\pi(t)}{t^2\log t}dt + \int_2^x \frac{\pi(t)}{t^2\log^2 t}dt$$

($\pi(x) = 0$ for $x < 2$ as 2 is the smallest prime). The first term is $O(1/\log x) = o(1)$ (this only needs $\pi(x) := \sum_{p\le x} 1 \le x$); the third term is negligible w.r.t. the second, which by Chebyshev's Upper Estimate is of order

$$\int_2^x \frac{(t/\log t)}{t^2\log t}dt = \int_2^x \frac{dt}{t\log^2 t} = \int_{\log 2}^{\log x} du/u^2 \le \int_c^{\infty} du/u^2 < \infty.$$

So $\sum 1/(p\log p)$ converges. **[10]**

All seen (lectures)

Q2. (i). $n!+2, \ldots, n!+n$ is a string of $n-1$ consecutive composite numbers (each of $2, \ldots, n$ divides $n!$).
Alternative proof (seen in problems – HW Th. 5). If $2, 3, \ldots, p$ are all the primes up to p, then all numbers n up to p are divisible by at least one of these primes, p' say, by FTA: $p'|n$, $n = p'r'$, say. So if $q := 2.3.5 \ldots p = \prod p'$, then all the $p-1$ numbers $q+2, q+3, q+4, \ldots, q+p$ are composite: for each is of the form $q+n = q+p'r' = p'.\prod p'' + p'r' = p'(r' + \prod p'')$, where the product is over the primes other than p in the prime-power factorisation of n (counted with multiplicity), and any repetitions of p. So this string of $p-1$ consecutive numbers forms (part of) a gap between primes. There are arbitrarily large p (Euclid), so arbitrarily long gaps between primes. **[6]**
(ii). Write

$$N := (2^2.3.5. \ldots p) - 1,$$

the product being over the primes up to p. Then 4 divides the product, so $N = 4n+3$ for some n (residue $3 = -1 \bmod 4$). N is not divisible by any of the primes up to p. Since a product of primes with residue 1 has residue 1, N must have a prime factor with residue 3 mod 4, $q = q(p)$ say. Discarding any p that give a q already encountered, as p runs through the primes we obtain infinitely many primes q of the form $4n+3$. **[6]**
(iii) Write

$$N := (2.3, 5. \ldots p) - 1,$$

the product again being over the primes up to p. Then 6 divides the product, so $N = 6n+5$ for some n (residue $5 = -1 \bmod 6$). As $6n, 6n+2, 6n+3, 6n+4$ are composite, the only candidates for primes have residue 1 or 5 mod 6. Since a product of primes with residue 1 has residue 1, N must have a prime factor with residue 5 mod 6, $q = q(p)$ say. Discarding any p that give a q already encountered, as p runs through the primes we obtain infinitely many primes q of the form $6n+5$. **[6]**
(iv) Bertand's postulate is that there for every natural number n there is a prime p with $n < p \leq 2n$; equivalently, if p_r is the rth prime, $p_{r+1} < 2p_r$. This was proved by Erdös in 1932, by elementary means. **[2]**

All seen (problems), except (iii) (similar to (ii)). (Parts (ii) and (iii) follow from Dirichlet's Theorem(s) on primes in arithmetic progressions – stated but not proved.)

Q3. (i) The class has seen two proofs:
(a) The alternating zeta function (Dirichlet eta function) $\eta(\sigma) := \sum_1^\infty (-)^{n-1}/n^\sigma$ converges for $\sigma > 0$ by the Alternating Series Test. So the half-plane of convergence of the Dirichlet series is $\sigma > 0$. As $\eta(s) = \sum_{odd} 1/n^s - \sum_{even} 1/n^s = \sum_o - \sum_e$ and $\zeta(s) = \sum_o + \sum_e$, subtraction gives $\eta(s) - \zeta(s) = -2\sum_e = -2\sum_1^\infty 1/(2n)^s = -2.2^{-s}\zeta(s)$: $\zeta(s) = \eta(s)/(1-2^{1-s}) = \eta(s)/(1-e^{-(1-s)\log 2})$. As $\eta(1) = \log 2$ from Abel's continuity theorem, this gives the analytic continuation of ζ to $\sigma > 0$, where it has a simple pole at 1 of residue 1.
(The other zeros of $1 - 2^{1-s}$, at $s = 1 + 2\pi ni/\log 2$ for n a non-zero integer, are cancelled by zeros of η, but this is not asked.)
(b) Euler's summation formula for $f(x) = 1/x^s$ gives (as $\sum_1^\infty 1/n^s = \zeta(s)$ and $\int_1^\infty dx/x^s = 1/(s-1)$)

$$\zeta(s) = \frac{1}{s-1} + 1 - s\int_1^\infty \frac{x - [x]}{x^{s+1}} dx,$$

and the integral is holomorphic for $\sigma > 0$. **[10]**
(ii) The class has seen four proofs: (i) using Complex Analysis to sum the series (M2PM3); (ii) from the product for sine (M2PM3); (iii) by calculus (LeVeque I, Ex. 6 p.122; Problems); (iv) by Fourier series (Problems), below.

Write a_n for the Fourier cosine coefficients of $|x|$ on $[-\pi, \pi]$ ($|.|$ is even, so we do not need sine terms). Then

$$\frac{1}{2}a_0 = \frac{1}{2}.\frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi}\int_0^\pi x dx = \frac{1}{\pi}[\frac{1}{2}x^2]_0^\pi = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos\ nx dx = \frac{2}{\pi}\int_0^\pi x \cos nx dx = \frac{2}{n\pi}\int_0^\pi x d \sin nx$$

$$= \frac{2[x \sin nx]_0^\pi}{n\pi} - \frac{2}{n\pi}\int_0^\pi \sin nx dx = \frac{2}{n^2\pi}[\cos nx]_0^\pi = \frac{2(\cos n\pi - 1)}{n^2\pi}$$

$$= \frac{2((-1)^n - 1)}{n^2\pi} = -\frac{4}{\pi n^2}$$

if n is odd, 0 if $n \neq 0$ is even. So

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^\infty \frac{\cos(2m-1)x}{(2m-1)^2}.$$

Putting $x = 0$ gives $0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{odd} 1/n^2$: $\sum_{odd} = \pi^2/8$. But $\zeta(2) = \sum_1^\infty 1/n^2 = \sum_{odd} + \sum_{even} = \sum_{odd} + \frac{1}{4}\zeta(2)$: $\frac{3}{4}\zeta(2) = \pi^2/8$, $\zeta(2) = \pi^2/6$. **[10]**

Seen: (i) (lectures), (ii) (problems).

Q4. (i) The Dirichlet convolution $(a * b)(n) := \sum_{d|n} a(d)b(n/d)$ corresponds to multiplication of the Dirichlet series for a and b. In the case of u, where $u(n) := 1$ for all $n = 1, 2, \ldots$, the Dirichlet series is $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$, by definition. Since

$$(u * u)(n) = \sum_{d|n} u(d)u(n/d) = \sum_{d|n} 1 = d(n),$$

this shows that the divisor function d has Dirichlet series $\zeta(s)^2$. **[5]**

(ii) **Theorem**. If d_n is the number of divisors of n,

$$\sum_{n \le x} d_n = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. Take $a_n = b_n = 1$ (so $(a * b)_n = d_n$, by (i)), $y = \sqrt{x}$: as $A(x) = B(x) = [x]$, Dirichlet's Hyperbola Identity gives

$$\sum_{n \le x} d_n = \sum_{j \le \sqrt{x}} [x/j] + \sum_{k \le \sqrt{x}} [x/k] - [\sqrt{x}][\sqrt{x}] = 2 \sum_{j \le \sqrt{x}} [x/j] - [\sqrt{x}][\sqrt{x}]. \quad \textbf{[5]}$$

In each [.] on RHS, write [.] = . − {.}. Each fractional part $\{.\} \in [0, 1)$, so

$$\sum_{n \le x} d_n = 2 \sum_{j \le \sqrt{x}} x/j + O(\sqrt{x}) - x + O(\sqrt{x}),$$

as $(\sqrt{x} + O(1))^2 = x + O(\sqrt{x})$. But (as in Lecture 3, Ch. I. §4, The Integral Test and Euler's constant),

$$\sum_{j \le \sqrt{x}} 1/j = \log \sqrt{x} + \gamma + O(1/\sqrt{x}) = \frac{1}{2} \log\ x + \gamma + O(1/\sqrt{x}).$$

So (as in Lecture 15, II.9)

$$\sum_{n \le x} d_n = 2x(\log \sqrt{x} + \gamma + O(1/\sqrt{x})) - x + O(\sqrt{x}) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

[10]

Seen (lectures).

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