m3pm16l12.tex
Lecture 12. 7.2.2013

$\log \zeta(s)$

Theorem. Write $c_n := 1/m$ if $n = p^m$ is a prime power, 0 otherwise. Then

$$H(s) := \sum_1^\infty c_n/n^s = \log \zeta(s) \qquad (Re\ s > 1).$$

Proof. $\zeta(s) = \prod 1/(1-p^{-s})$, so $\log(\zeta(s)) = -\sum_p \log(1-p^{-s})$. As $-\log(1-z) = \sum_1^\infty z^m/m$ for $|z| < 1$,

$$\log \zeta(s) = \sum_p \sum_1^\infty \frac{1}{m}.1/p^{ms} = \sum_{m=1}^\infty \frac{1}{m} \sum_{n=p^m} 1/n^s = \sum_{n=1}^\infty c_n/n^s = H(s),$$

in the half-plane of absolute convergence $Re\ s > 1$, where changing of order of summation is justified. The function $\log \zeta(s)$ is holomorphic on $\sigma > 1$ as $\zeta(s)$ is holomorphic and nonzero there. //

The Von Mangoldt Function $\Lambda(n)$, 1899.

$\Lambda(n) := \log(p)$ if $n = p^m$ is a prime power, 0 otherwise. Differentiating the result above gives

$$\zeta'(s)/\zeta(s) = -\sum_1^\infty c_n \log n/n^s.$$

But

$$c_n \log n = \frac{1}{m} \cdot m \log p = \log p \text{ if } n = p^m \text{ is a prime power },0 \text{ otherwise.}$$

So

$$-\zeta'(s)/\zeta(s) = \textstyle\sum_1^\infty \Lambda(n)/n^s \qquad (Re\ s > 1). \qquad (\Lambda, -\zeta'/\zeta)$$

Corollary. With $l(n) := \log(n)$, $\Lambda * u = l, l * \mu = \Lambda$.

Proof. $\Lambda(1) = l(1) = 0$. For $n > 1, n = p_1^{r_1}...p_k^{r_k}$, say. Then $(\Lambda * u)(n) = \sum_{i|n} \Lambda(i)$. The divisors i of n are $i = p_1^{s_1}...p_1^{s_k}, 0 \leq s_j \leq r_j$. Those with $\Lambda(i) \neq 0$ are only those with $i = p_j^{s_j}$, each of which has $\Lambda(i) = \log p_j$. There

are k elements in this sum (ignoring the case of $i = 1$).
$\sum_{i|n} \Lambda(i) = \sum_{j=1}^{k} r_j \log p_j = \log \prod_j p_j^{r_j} = \log n = l(n)$. So $\Lambda * u = l$, and then $l * \mu = \Lambda$ follows by Möbius inversion. //

Write
$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$
Then also
$$\psi(x) = \sum_{p^m \leq x} \log p.$$
As the highest power m with $p^m \leq x$ is $m = \left[\frac{\log x}{\log p}\right]$:
$$\psi(x) = \sum_{p \leq x} \left[\frac{\log x}{\log p}\right] \log p.$$
By Abel summation (I.3), $-\zeta'(s)/\zeta(s) = \sum_1^\infty \Lambda(n)/n^s$, and
$$\frac{\zeta'(s)}{\zeta(s)} = -s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx \qquad (\mathit{Re}\ s > 1).$$
We shall see that PNT ($\pi(x) \sim li(x) = x/\log x$) $\Leftrightarrow \psi(x) \sim x$, and this is the way we will prove the Prime Number Theorem.

7. Mertens' Theorems

Theorem (HW Th. 424). For $x > 1, \sum_{n \leq x} \Lambda(n)/n = \log x + r(x)$, with $|r(\cdot)| \leq 2$.

Proof. By II.5, $\left|\sum_1^N \mu(n)/n\right| \leq 1$. Write $S(x) := \sum_{n \leq x} \log n$. As $\Lambda * u = l$,
$$S(x) = \sum_{n \leq x} (\Lambda * u)(n) = \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] \qquad \text{(with [.] the integer part)}$$
$$= x \sum_{n \leq x} \frac{\Lambda(n)}{n} - a(x), \qquad \text{where} \qquad a(x) := \sum_{n \leq x} -\Lambda(n)\{x/n\},$$
with $\{.\}$ the fractional part. Then
$$0 \leq a(x) = \sum_{n \leq x} \Lambda(u)\{x/n\} \leq \sum_{n \leq x} \Lambda(u) = \psi(x) \leq 2x,$$
by Chebyshev's Upper Estimate (III.2).