m3pm16l14.tex
Lecture 14. 12.2.2013

Theorem(Merten's Formula, HW Th 929).

$$\prod_{p \le x} (1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x} \qquad (x \to \infty).$$

Proof. Write $\sum := \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right)$, which is convergent. By Merten's Second Theorem and the Constants Lemma (from the Website),

$$\sum_{p \le x} \frac{1}{p} = \log\log x + C_1 + o(1) = \log\log x + \gamma + \Sigma + o(1).$$

Now,

$$\sum_{p \le x} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = \Sigma + o(1),$$

from the definition of Σ. Subtracting:

$$\sum_{p \le x} \log\left(1 - \frac{1}{p}\right) = -\log\log x - \gamma + o(1).$$

That is,

$$\log\left[\prod_{p \le x}\left(1 - \frac{1}{p}\right)\right] = \log\left[\frac{e^{-\gamma}}{\log x}\right] + o(1).$$

So

$$\log\left[\prod_{p \le x}(1 - \frac{1}{p}) / \frac{e^{-\gamma}}{\log x}\right] \to 0 \qquad (x \to \infty).$$

So [...] $\to 1$, i.e.

$$\prod_{p \le x}\left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x}. \qquad //$$

8. Prime Divisor Functions

Recall the following arithmetic functions: $d(n) :=$ #divisors of n; $\omega(n) :=$ # distinct prime divisors n; $\Omega(n) :=$ # prime divisors n (counted with multiplicity). So if $n = p_1^{r_1} ... p_k^{r_k}$, we have

$$d(n) = \prod_1^k (1 + r_j), \qquad \omega(n) = k, \qquad \Omega(n) = \sum_1^k r_j.$$

Theorem. (i) $\sum_{n \leq x} \omega(n) = x \log\log x + C_1 x + O(1/\log x)$,
(ii) $\sum_{n \leq x} \Omega(n) = x \log\log x + C_2 x + O(1/\log x)$,
where as above

$$C_1 = \gamma + \sum, \qquad C_2 = C_1 + \sum_p \frac{1}{p(p-1)} = C_1 + S,$$

say.
Proof. (i) $\sum_{n \leq x} \omega(n)$ is the number of pairs (p, n) with $p|n$ and $n \leq x$. For fixed p, the number of such pairs is equal to the number of multiples $rp \leq x$, i.e. $[x/p]$. So

$$\sum_{n \leq x} \omega(n) = \sum_{p \leq x} \left(\frac{x}{p} - \left\{ \frac{x}{p} \right\} \right).$$

By the first theorem of II.7,

$$\sum_{p \leq x} \frac{x}{p} = x \sum_{p \leq x} \frac{1}{p} = x \log\log x + C_1 x + O(x/\log x).$$

Also

$$0 \leq \sum_{p \leq x} \{x/p\} < \sum_{p \leq x} 1 = \pi(x) = O(x/\log x),$$

by Chebyshev's Upper Estimate (III.2). Combining gives (i).
(ii) Similarly. //

Note. This theorem gives us:

$$\frac{1}{x} \sum_{n \leq x} \omega(n) = \log\log x + C_1 + O(1/\log x);$$
$$\frac{1}{x} \sum_{n \leq x} \Omega(n) = \log\log x + C_2 + O(1/\log x).$$

In Probabilistic Number Theory (III.9, III.10.1), we think of the LHS as the 'mean value' of ω, Ω. This says that both $\sim \log\log x$. This result, due to Hardy and Ramanujan (1917), corresponds to the Law of Large Numbers (LLN), thinking of ω, Ω as random and divisibility by distinct primes as independent events. We shall prove the Prime Number Theorem (PNT), and extensions of it, counting $n \leq x$ with k prime factors. This will correspond to the relevant Central Limit Theorem (CLT), extending the Law of Large Numbers.