m3pm16l15.tex
Lecture 15. 14.2.2013

9. Dirichlet's Hyperbola Identity (DHI)

Theorem (DHI). If $1 < y < x$,

$$\sum_{n \le x} (a * b)(n) = \sum_{j \le y} a(j)B(x/j) + \sum_{k \le x/y} b(k)A(x/k) - A(y)B(x/y).$$

Proof. LHS $= S := \sum_{jk \le x} a_j b_k$, as in II.3. Write S_1 for the sum of all such terms with $j \le y$, S_2 that of all terms with $k \le x/y$. As in II.3,

$$S_1 = \sum\nolimits_{jk \le x, j \le y} a_j b_k = \sum\nolimits_{j \le y} a_j \sum\nolimits_{k \le x/j} b_k = \sum\nolimits_{j \le y} a_j B(x/k),$$

the first sum on RHS, and similarly

$$S_2 = \sum\nolimits_{jk \le x, k \le x/y} a_j b_k = \sum\nolimits_{k \le x/y} b_k \sum\nolimits_{j \le x/k} a_j = \sum\nolimits_{k \le x/y} b_k A(x/k),$$

the second sum on RHS. Now $S_1 + S_2$ counts all terms, but counts twice those with both $j \le y$ and $k \le x/y$. The sum of these terms is $A(y)B(x/y)$. So subtracting this 'corrects the count', and gives the result. //

Theorem. If d_n is the number of divisors of n,

$$\sum_{n \le x} d_n = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. Take $a_n = b_n = 1$ (so $(a * b)_n = d_n$, by (i)), $y = \sqrt{x}$: as $A(x) = B(x) = [x]$, Dirichlet's Hyperbola Identity gives

$$\sum_{n \le x} d_n = \sum_{j \le \sqrt{x}} [x/j] + \sum_{k \le \sqrt{x}} [x/k] - [\sqrt{x}][\sqrt{x}] = 2 \sum_{j \le \sqrt{x}} [x/j] - [\sqrt{x}][\sqrt{x}].$$

In each [.] on RHS, write $[.] = . - \{.\}$. Each fractional part $\{.\} \in [0, 1)$, so

$$\sum_{n \le x} d_n = 2 \sum_{j \le \sqrt{x}} x/j + O(\sqrt{x}) - x + O(\sqrt{x}) = 2x \sum_{j \le \sqrt{x}} 1/j - x + O(\sqrt{x}),$$

as $(\sqrt{x} + O(1))^2 = x + O(\sqrt{x})$. But as in L3, I.4,

$$\sum\nolimits_{j \le \sqrt{x}} 1/j = \log \sqrt{x} + \gamma + O(1/\sqrt{x}) = \frac{1}{2} \log\ x + \gamma + O(1/\sqrt{x}).$$

So

$$\sum_{n \le x} d_n = 2x(\log \sqrt{x} + \gamma + O(1/\sqrt{x})) - x + O(\sqrt{x}) = x \log x + (2\gamma - 1)x + O(\sqrt{x}). \ //$$

III. THE PRIME NUMBER THEOREM AND ITS RELATIVES

§1. The Prime Number Theorem (PNT)

PNT states that

$$\pi(x) := \sum\nolimits_{p \leq x} 1 \sum li(x) := \int_2^x dt/\log t \sim x/\log x \qquad (x \to \infty) \qquad (PNT)$$

This was conjectured on numerical grounds by GAUSS (c. 1799; letter of 1848) and A. M. LEGENDRE (1752-1833; in 1798, *Essai sur la Théorie des Nombres*).

In 1737 L. EULER (1707-1783) found his Euler product, linking the primes to $\sum_{n=1}^{\infty} 1/n^{\sigma}$ for real σ (later the Riemann zeta function).

In 1850-51, P. L. CHEBYSHEV (= TCHEBYSHEV, etc., 1821-1894) made two great strides (III.2):
(i) $\pi(x) \asymp x/\log x$, i.e. $C_1 x/\log x \leq \pi(x) \leq C_2 x/\log x$ for some $0 < C_1 \leq C_2 < \infty$ and all $x \geq X$;
(ii)

$$\liminf \pi(x)/\frac{x}{\log x} \leq 1 \leq \limsup \pi(x)/\frac{x}{\log x}$$

– so if the limit exists (which we shall prove!), it must be 1.

In 1859 B. RIEMANN (1626-66) studied

$$\zeta(s) := \sum\nolimits_{n=1}^{\infty} 1/n^s \qquad (s \in \mathbf{C})$$

using Complex Analysis (M2PM3), then still fairly new, developed by A. L. CAUCHY (1789-1857), 1825-29. He showed the critical relevance of the *zeros of* $\zeta(s)$ to the *distribution of primes*. We shall show that:
(i) ζ can be continued analytically from $Re\ s > 1$ to the whole complex plane $\mathbf{C}$, where it is holomorphic except for a simple pole at 1 of residue 1 (III.3);
(ii) The only zeros of ζ outside the *critical strip*

$$0 < \sigma = Re\ s < 1$$

are the so-called *trivial zeros* $-2, -4, \ldots, -2n, \ldots$ (trivial in that they follow from the *functional equation* for ζ – see III.3);
(iii) PNT is closely linked to non-vanishing of ζ on the 1-line (III.4):

$$\zeta(1 + it) \neq 0.$$

Indeed, PNT is equivalent to this (see III.10.4 for this and other such equivalences).