m3pm16l17.tex

Lecture 17. 19.2.2013

Recall (II.6): $\psi(x) := \sum_{n \le x} \Lambda(n) = \sum_{p^m \le x} \log p = \sum_{p \le x} [\log x / \log p] \log p$ (Chebyshev's notation for ψ, Λ the von Mangoldt function),

$$\zeta'(s)/\zeta(s) = -\sum_1^\infty \Lambda(n)/n^s = -s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx \qquad (Re\ s > 1).$$

As $\Lambda(n) = \log n$ if $n = p^m$, 0 otherwise, if $p_1, ..., p_n$ are the primes $\le x$, and k_j for the largest k with $p_j^k \le x$, then each $p_j^k (1 \le k \le k_j)$ contributes $\log p_j$ to $\psi(x)$, so $\psi(x) = k_1 \log p_1 + ... + k_n \log p_n$. So:

Proposition 1. $\psi(x) \le \pi(x) \log x$.

Proof. $n = \pi(x)$ into the above, and then $k_j \log p_j \le \log x$ as $p_j^{k_j} \le x$. //

Recall: **ENT1**. If $p|ab$, then $p|a$ or $p|b$.
ENT2. If m, n are coprime, and both divide a, then $mn|a$.

Theorem 2 (Chebyshev's Upper Estimates).
(i) $\theta(x) \le (\log 4) x$.
(ii) $\pi(x) \le (\log 4) li(x) + 4$.

Proof. Fix n, and write $N := \binom{2n+1}{n} = (2n+1)(2n)...(n+2)/n!$ Now, $N = \binom{2n+1}{n} = \binom{2n+1}{n+1}$, two terms from the binomial expansion of $(1+1)^{2n+1} = 2^{2n+1}$. So $2N \le 2^{2n+1} : N < 4^n$, giving $\log N < n \log 4$.

Let $p_{k+1}, ...p_m$ be the primes with $n+2 \le p \le 2n+1$, so $\sum_{k+1}^m \log p_j = \theta(2n+1) - \theta(n+1)$. By (ENT1), no such p divides $n!$, but each divides $(2n+1)...(n+2) = n!N$. So by (ENT1), each divides N, and by (ENT2) their product divides N, so is $\le N$. So

$$\theta(2n+1) - \theta(n+1) = \log(p_{k+1}...p_m) \le \log N < n \log 4. \qquad (*)$$

We now show by induction that $\theta(n) \le n \log 4 \qquad (n \ge 2)$.
The induction starts, as $\theta(2) = \log 2 \le 2 \log 4$.
Assume that the condition holds for all $k \le 2n$, for $n \ge 1$.
Then in particular, $\theta(n+1) \le (n+1) \log 4$, but we have by $(*)$:

$$\theta(2n+1) \le (2n+1) \log 4.$$

Also, $\theta(2n+2) = \theta(2n+1)$, as $2n+2$ is not prime. So

$$\theta(2n+2) \leq 2n+1)\log 4 \leq (2n+2)\log 4,$$

completing the induction. Part (ii) follows from (i), as $\alpha \log 4 = 4$. //

Corollary 1. $\pi(x) \leq C_1 x/\log 2$ for $x \geq 2$ and some constant $c_1 \leq 3.1\log 4$.

Proof. By the Theorem and Problems 1. //

Corollary 2. $\psi(x) \leq C_1 x$.

Proof. $\psi(x) \leq \pi(x)\log x$ and then apply Corollary 1. //

Proposition 2. For m the largest integer with $2^m \leq x$, $\psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + ... + \theta(x^{1/m})$. //

Proof. See J p. 76.

Proposition 3. (i) $\psi(x) - \theta(x) \leq 6\sqrt{x}$ for $x > 1$.
(ii) $\forall \epsilon > 0, \psi(x) \leq (\log 4 + \epsilon)x$ for large enough x.

Proof. For (i), use the result above, and as $\theta(\cdot)$ is increasing:

$$\psi(x) - \theta(x) \leq \theta(\sqrt{x}) + m\theta(x^{1/3}) \qquad (m \leq \log x/\log 2).$$

So by Chebyshev's Upper Estimate for θ, $\psi(x) - \theta(x) \leq x^{1/2}\log 4 + 2x^{1/3}\log x$. But $x^{1/3}\log x \leq \frac{6}{e}x^{1/2}$ (check: the maximum of $\log(x)/x^{\alpha}$ is $1/(\alpha e)$). So $\psi(x) - \theta(x) \leq (\log 4 + 12/e)x^{1/2} < 6x^{1/2}$, giving (i). For (ii), use (i) and the fact that $\theta(x) \leq (\log 4)x$. //

Corollary 3. $(\psi(x) - \theta(x))/x \to 0$ $(x \to \infty)$.

So if either of $\psi(x)/x, \theta(x)/x$ has a limit, both do and they are the same. Now PNT is $\pi(x) \sim li(x) \sim x/\log x$. So ($c = C$ in the first Chebyshev Theorem above) gives:

Theorem (Equivalence Theorem). The following are equivalent:
(i) PNT: $\pi(x) \sim li(x) \sim x/\log x$; (ii) $\psi(x) \sim x$; (iii) $\theta(x) \sim x$.