m3pm16l18.tex
Lecture 18. 21.2.2013

Cor. 4. $\psi(x) < 2x$ $(x > 1)$.

Proof (sketch – see J p.77 for details).

$$\frac{\psi(x)}{x} \le \theta(x)x + \frac{6}{\sqrt{x}} \le \log 4 + \frac{6}{\sqrt{x}}, < 2 \qquad (x > 1).$$

Powers of primes. Write π^* for the prime-power counting function, $\pi^*(x) := \sum_{p^m \le x} 1$. Then as above, we find

$$\pi^*(x) = \pi(x) + \pi(\sqrt{x}) + \ldots + \pi(x^{1/m}),$$

with m the largest integer with $2^m \le x$, and

$$\pi^*(x) - pi(x) \le 12C\sqrt{x}/\log x \qquad (x \ge 2),$$

with C s.t. $\pi(x) \le Cx/\log x$ $(x \ge 2)$. For details, see [J] p.78-79.

Chebyshev's Lower Estimates.

Write $\nu := e_1 - 2e_2$: $\nu(1) = 1$, $\nu(2) = -2$, $\nu(n) = 0$ for $n \ge 2$. Then

$$(u*\nu)(x) = \sum_{i|n} \nu(i).1 = 1 \quad (n \text{ odd } : i = 1 \text{ only }), \quad -1 \quad (n \text{ even } : i = 1, 2).$$

Let $E(x) := \sum_{n \le x}(u * \nu)(n)$. Then $E(x) = 1$ if $[x]$ is odd, 0 if $[x]$ is even.
LEMMA 1.

$$\sum_{j \le x} \Lambda(j)E(x/j) = \sum_{k \le x} \log k - 2 \sum_{k \le x/2} \log k \quad (x \ge 2).$$

Proof. By the Lemma of II.3 (sum of a convolution),

$$\begin{aligned}
\sum_{j \le x} \Lambda(j)E(x/j) &= \sum_{j \le x} [\Lambda * (u * \nu)](j) \qquad (\text{Lemma: } E \text{ sum-function of } u * \nu) \\
&= \sum_{j \le x} (\ell * \nu)(j) \qquad (\Lambda * \nu = \ell) \\
&= \sum_{j \le x} \nu(j) \sum_{k \le x/j} \log k \qquad (\ell = \log; \text{ Lemma again}) \\
&= \sum_{k \le x} \log k - 2 \sum_{k \le x/2} \log k \quad (x \ge 2). \quad //
\end{aligned}$$

LEMMA 2.

$$\psi(2n) \geq \log \binom{2n}{n}.$$

Proof. Take $x = 2n$ in the Lemma, and let S be the sum on the left. As each $E(.) \leq 1$,

$$S \leq \sum_{j \leq 2n} \Lambda(j) = \psi(2n).$$

But

$$\sum_{k \leq x} \log k - 2 \sum_{k \leq x/2} \log k = \sum_{k=n+1}^{2n} \log k - \sum_{k=1}^{n} \log k + \log\Big(\frac{(n+1)(n+2)\dots(2n)}{1.2.\dots n}\Big) = \log \binom{2n}{n}. \quad //$$

THEOREM 3 (Chebyshev's Lower Estimates). For $\epsilon > 0$ and x sufficiently large,
(i) $\psi(x) \geq (\log 2 - \epsilon)x$;
(ii) $\theta(x) \geq (\log 2 - \epsilon)x$;
(iii) $\pi(x) \geq (\log 2 - \epsilon) li(x)$.

Proof. (i) Let $N := \binom{2n}{n}$ as above. This is the largest of the $2n+1$ terms in the binomial expansion of $(1+1)^{2n}$ (by Pascal's triangle), so $2^{2n} \leq (2n+1)N$. So by the Lemma above,

$$\psi(2n) \geq \log N \geq 2n \log 2 - \log(2n+1).$$

Given x, take n with $2n \leq x < 2n+2$. Then by above

$$\psi(x) \geq (x-2)\log 2 - \log(x+1),$$

giving (i).
(ii) This follows from (i) as $(\psi(x) - \theta(x))/x \to 0$ (Cor. above).
(iii) This follows from (ii) by the first Theorem of this section. //

Cor. 5. $\pi(x) \geq (\log 2 - \epsilon)x/\log x$.

Proof. $\psi(x) \leq \pi(x) \log x$ (first Prop. of this section and (i). //

In 1849-51 Chebyshev proved that if $\pi(x)/li(x)$ has a limit, it must be 1 (L, 11-29, esp. 16). We omit the proof. In 1851, Chebyshev also proved *Bertrand's postulate* of 1845: for any $n \geq 2$ there is a prime p between n and $2n$; see Problems and Solutions 8 for Erdös' elementary proof of 1932.