m3pm16l20.tex
Lecture 20. 26.2.2013

4. Non-vanishing on the 1-line: $\zeta(1+it) \neq 0$.

Lemma. $3 + 4\cos\theta + \cos 2\theta \geq 0$.

Proof. $3 + 4\cos\theta + \cos 2\theta = 2 + 4\cos\theta + 2\cos^2\theta = 2(1+\cos\theta)^2$. //

Prop. If all $a_n \geq 0$ and the Dirichlet series $f(s) := \sum_1^\infty a_n/n^s$ converges for $Re\ s = \sigma > \sigma_0$, then

$$3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) \geq 0 \qquad (\sigma > \sigma_0).$$

Proof.

$$3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) = \sum_1^\infty \frac{a_n}{n^\sigma}(3 + 4n^{-it} + n^{-2it}).$$

If $\theta_n := t\log n$, $Re(3 + 4n^{-it} + n^{-2it} = 3 + 4\cos\theta_n + \cos 2\theta_n \geq 0$. //

Corollary. For $\sigma > 1$ and all t,

$$H(\sigma) := \zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq 1.$$

Proof. By II.6, $\log\zeta(s)$ has a Dirichlet series with non-negative coefficients, $\log\zeta(s) = f(s) = \sum_1^\infty a_n/n^s$ for $a_n \geq 0$. By the Proposition, $3f(\sigma) + 4Ref(\sigma+it) + Ref(\sigma+2it) \geq 0$. So ($\log z = \log(re^{i\theta}) = \log r + i\theta$, so $Re\log z = \log r = \log|z|$)

$$3\log\zeta(\theta) + 4\log|\zeta(\theta+it)| + \log|\zeta(\theta+2it)| \geq 0.$$

Exponentiating gives the result. //

Theorem. $\zeta(1+it) \neq 0$ for $t \neq 0$.

Proof (by contradiction). If not, $\zeta(1+it) = 0$ for some $t \neq 0$. Then differentiating from first principles,

$$\frac{\zeta(\sigma+it) - \zeta(1+it)}{(\sigma+it)-(1+it)} = \frac{\zeta(\sigma+it)}{\sigma-1} \to \zeta'(1+it) \qquad (\sigma \downarrow 1),$$

as ζ is holomorphic at $1+it$. In the Corollary,

$$H(\sigma) = [(\sigma-1)\zeta(\sigma)]^3 \left(\frac{|\zeta(\sigma+it)|}{\sigma-1}\right)^4 [(\sigma-1)|\zeta(\sigma+2it)|].$$

Now $(\sigma-1)\zeta(\theta) \to 1$ $(\sigma \downarrow 1)$ (ζ has a simple pole of residue 1 at 1). So $[...]^3 \to 1$; $(...)^4 \to (\zeta'(1+it))^4$ by above; $|\zeta(\sigma+2it)| \to \zeta(1+2it)$. Combining, $H(\sigma) \to 0$ as $\sigma \to 1$, contradicting the Corollary above. //

Note. 1. The critical term in the proof above is the factor $\sigma - 1$ in the last [...] (available because of the "3, 4, 1" coefficients in the Lemma (see below).
2. $\zeta(1+it) \neq 0$ is essentially equivalent to the PNT, below.

Recall: from the Euler product, $\zeta \neq 0$ to the *right* of the 1-line; by the Theorem, $\zeta \neq 0$ *on* the 1-line. We now extend the zero-free region of ζ to the *left* of the 1-line and into the *critical strip* of $0 \leq \sigma \leq 1$. It suffices to consider $t > 0$, as $|\zeta(\sigma - it)| = |\zeta(\sigma+it)|$ (since $n^{-s} = e^{-it\log n}/n^{\sigma}$).

Theorem. For $0 < a < b$, $\exists \delta > 0$ such that $\zeta(\sigma+it) \neq 0$ in $1-\delta \leq \sigma \leq 1$, $a \leq t \leq b$ (a rectangle *inside* the critical strip).

Proof. If not, for each n there exists some $s_n = \sigma_n + it_n$ with

$$1 - 1/n \leq \sigma_n \leq 1, \ a \leq t_n \leq b, \ \zeta(s_n) = 0.$$

As t_n is an infinite sequence in $[a,b]$, which is compact, it has a convergent subsequence t_{n_k} (Bolzano-Weierstrass Th.), going to t_0, say. Then $\sigma_n \to 1$, so $s_{n_k} \to 1 + it_0$. So $\zeta(s_{n_k}) \to \zeta(1+it_0)$ by the continuity of ζ, and this is non-zero by the Theorem above. But each $\zeta(s_n) = 0$, so $\zeta(s_{n_k}) = 0 \to 0$, a contradiction. //

Note. 1. The method of proof above, resting on the Lemma, is due to Hadamard in his original proof of PNT in 1896. It is clear and efficient, but seems unmotivated (or like a 'trick'). For an approach which both seems more natural and is more general (non-vanishing of Dirichlet L-series, rather than just the zeta function), see Newman [N], VI: *A "natural" proof of the non-vanishing of L-series.*
2. $\zeta(1+it) \neq 0$ is exactly what is needed to apply the most important Tauberian theorem, Wiener's Tauberian theorem; see III.10.3.