m3pm16l21.tex.tex
Lecture 21. 28.2.2013.

5. Further Fourier Analysis

We shall need several facts about Fourier analysis. For proofs, see e.g. [Kat], [AL]. This suffices for our proof of PNT (without remainder).

Fourier Integral Theorem (FIT).

Recall (I.6) the Riemann-Lebesgue lemma: if $f \in L_1(\mathbb{R})$, with Fourier transform

$$\hat{\phi}(\lambda) := \int_{-\infty}^{\infty} e^{i\lambda t}\phi(t)dt,$$

then $\hat{\phi}(\lambda) \to 0$ as $\lambda \to \infty$. Usually $\hat{f} \in L_1$ does not decay fast enough at infinity to be integrable (to be in L_1), but if it does we can apply the Fourier transform to $\hat{f}$. It turns out that it is convenient to make two changes in doing so: to change $e^{\cdot}$ to $e^{-\cdot}$, and to introduce a factor $1/(2\pi)$. This gives the *inverse Fourier transform*, defined for $g \in L_1$ by

$$g^{\vee}(\lambda) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}g(x)dx.$$

If we do the operations $f \mapsto f^{\wedge}$ – (direct) Fourier transform – and then $g \mapsto g^{\vee}$ – inverse Fourier transform – we get back to where we started. This is a form of the *Fourier Integral Theorem*: if $f, \hat{f} \in L_1$,

$$f^{\wedge\ \vee} = f. \qquad (FIT)$$

Even if $\hat{f} \notin L_1$, one can still get f as a limit of an integral involving $\hat{f}(\lambda)e^{-i\lambda x}$ in the integrand (and the transform of the Fejér kernel below) – see e.g. [Kat], VI.1.11. By Lebesgue's dominated and/or monotone convergence theorems from Measure Theory, one can take the limit if $\hat{f} \in L_1$ and get (FIT).

The FIT also holds for f a *Schwartz function* (smooth – i.e., C^{∞} – functions all of whose derivatives decay at $\pm\infty$ faster than powers) – see [AL].

Convolution.

If $f, g \in L_1$, the *convolution* of f and g is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

We quote: $f * g \in L_1$, and

$$(f * g)^{\wedge} = f^{\wedge}g^{\wedge} :$$

the Fourier transform of the convolution is the product of the Fourier transforms.

Fejér kernel.

The *Fejér kernel* ([Kat], VI.1) is defined by

$$K_\lambda(x) := \lambda K(\lambda x) \qquad (\lambda > 0),$$

where

$$K(x) := \frac{1}{2\pi}\Big(\frac{\sin\ \frac{1}{2}x}{\frac{1}{2}x}\Big)^2.$$

Then

$$\hat{K}_\lambda(t) = (1 - |t|/\lambda)_+,$$

and (check by calculus)

$$K(x) = \frac{1}{2\pi}\int_{-1}^{1}(1 - |\lambda|)e^{-i\lambda x}d\lambda.$$

The thing to note here is that K is entire, and $\hat{K}$ has compact support (so in particular, $\hat{K} \in L_1$, so (FIT) holds for K – as the formula above tells us).

Convolution with the Fejér kernel is often useful. If $f \in L_1$, then so is $f * K$, whose Fourier transform $\hat{f}\hat{K}$ has compact support (as $\hat{K}$ does), so is in L_1, so (FIT) applies to it. We shall use this in our proof of PNT.

Poisson Summation Formula.

We quote ([AL], or [Kat] VI.1.15) the *Poisson summation formula*:

$$2\pi\lambda \sum_{n=-\infty}^{\infty} f(2\pi\lambda n) = \sum_{n=-\infty}^{\infty} \hat{f}(n/\lambda). \qquad (PSF)$$

Examples.

Gaussian kernel. We quote:

$$\int_{-\infty}^{\infty} e^{ixt}\frac{e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} = e^{-\frac{1}{2}t^2}.$$

In probabilistic language, this says that the standard normal density $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$ has characteristic function (CF) $e^{-\frac{1}{2}t^2}$; see e.g. NHB, M2PM3, L1, L25. Analytically, this says that $e^{-\frac{1}{2}x^2}$ is essentially its own Fourier transform (indeed, it *is* its own Fourier transform if we split the factor $1/(2\pi)$ in FIT into $1/\sqrt{2\pi}$ in each of the direct and inverse Fourier transforms).