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Lecture 22. 5.3.2013.

Theta function.

If we use this in (PSF), we find after a change of variables that

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi/x}. \qquad (\theta)$$

This is one of Jacobi's identities for the *Jacobi theta function* (transformation under the modular group); see e.g. [WW] 21.51, or Apostol [A2], p.91, 141. It can be re-written as follows: if

$$\Psi(x) := \sum_{n=1}^{\infty} e^{-n^2\pi x},$$

then

$$2\Psi(x) + 1 = \frac{1}{\sqrt{x}}(2\Psi(1/x) + 1). \qquad (\theta)$$

We shall use this result in our proof of the functional equation for the Riemann zeta function.

Paley-Wiener theorem.

The relationship between the Fejér kernel K and its FT $\hat{K}$ is an instance of the *Paley-Wiener theorem* (see e.g. [Kat] VI.7.4). One says that an entire function f is of *exponential type* $a > 0$ if

$$f(z) = o(e^{a|z|}) \qquad (z \to \infty).$$

Then the Paley-Wiener theorem says that the following are equivalent:
(i) $f(z)$ is an entire function of exponential type a and its restriction $f(x)$ to $\mathbb{R}$ is in $L_2(\mathbb{R})$ (i.e. $f^2 \in L_1(\mathbb{R})$;
(ii) $\hat{f}$ has compact support $[-a, a]$.
Thus the growth of f at infinity is accurately tied to the support of its Fourier transform.

6. The Wiener-Ikehara theorem

The Wiener-Ikehara theorem is the prototypical complex Tauberian theorlem (see Handout 'Tauberian thorems'). We follow Korevaar [Kor], III.4. The proof goes back to Wiener in 1928 and 1932, Ikehara in 1931 and Bochner in 1933. Recall (Handout: Transforms) the Laplace-Stieltjes (LS) transform (LST). As usual, we use the *additive* form for proofs (Fourier transforms, on the line, Haar measure Lebesgue measure dx), but the *multiplicative* form for applications (Mellin transforms, on the half-line, Haar measure $x^{-1}dx = dx/x$). The proofs of this section are not examinable.

We use the Fejér kernel (III.5) K_λ $(0 < \lambda \to \infty)$. The kernel is non-negative; it is an *approximate identity* ($\hat{K}_\lambda \to 1$ as $\lambda \to \infty$), and $\hat{K}_\lambda$ has compact support. Also the Fourier Integral Theorem holds here:

$$K_\lambda^{\wedge\ \vee} = K_\lambda.$$

We use the *Heaviside function* H:

$$H(x) := 0 \qquad (x < 0), \qquad 1 \qquad (x \geq 0),$$

(unit jump function – probability density function of the constant 0), with LT $\hat{H}(z) = 1/z$: $\hat{H}(x+iy) = 1/(x+iy)$.

Proposition. If $\sigma(t) = 0$ $(t < 0)$, ≥ 0 $(t \geq 0)$,

$$F(z) := \hat{\sigma}(z) := \int_0^\infty \sigma(t) e^{-zt} dt \qquad (z = x + iy)$$

exists for $x < 0$, and as $x \downarrow 0$

$$G(z) := F(z) - A/z \to G(iy),$$

where $G(i.) \in L_1(-\lambda, \lambda)$ – then the integral

$$\int_{\mathbb{R}} K_\lambda(u-t)\sigma(t)dt = \int_{-\infty}^{\lambda u} \sigma(u - v/\lambda) K(v) dv$$

exists and

$$\to A \int_{\mathbb{R}} K(v)dv = A \qquad (u \to \infty).$$

Proof. $F(z) := \int_0^\infty \sigma(t) e^{-xt} e^{-iyt} dt$, so $\sigma(t)e^{-xt}$ has FT $F(z) = F(x+iy)$, while $K_\lambda(t)$ has FT $\hat{K}_\lambda(y)$. So their convolution

$$u \mapsto \int_{\mathbb{R}} K_\lambda(u-y)\sigma(t)e^{-xt}dt$$

has FT $\hat{K}_\lambda(y)F(x+iy)$. This has compact support, and is continuous, so (bounded and) integrable. So the Fourier Integral Theorem applies (III.5):

$$\int_{\mathbb{R}} K_\lambda(u-y)\sigma(t)e^{-xt}dt = \frac{1}{2\pi}\int \hat{K}_\lambda(y)F(x+iy)e^{iuy}dy = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \ldots.$$