m3pm16l23.tex

Lecture 23. 5.3.2013

Proof of the Proposition (continued)

By the above with H for σ:

$$\int_0^\infty K_\lambda(u-y)e^{-xt}dt = \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)\hat{H}(x+iy)e^{iuy}dy.$$

Write $G := F - AH$. The last two equations give

$$\int_0^\infty K_\lambda(u-t)\sigma(t)e^{-xt}dt = A\int_0^\infty K_\lambda(u-t)e^{-xt}dt + \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)G(x+iy)e^{iuy}dy.$$

We will let $x \downarrow 0$ here. As $K_\lambda \in L_1$, the first integral tends to $\int_0^\infty K_\lambda(u-t)dt$ by dominated convergence. For the second term on RHS: $K_\lambda \in L_1$, $\hat{K}_\lambda$ is continuous, and from the condition on F $G(x+iy) \to G(iy)$ $(x \downarrow 0)$, where $G(i.) \in L_1(-\lambda, \lambda)$. So the RHS above has a finite limit as $x \downarrow 0$. So the LHS does also. Since $K_\lambda(.)\sigma(.) \geq 0$, the LHS $\uparrow$ as $x \downarrow 0$. As the limit of the integrals exists, the limit is integrable by monotone convergence. So letting $x \downarrow 0$ gives

$$\int_0^\infty K_\lambda(u-t)\sigma(t)dt = A\int_0^\infty K_\lambda(u-t)dt + \frac{1}{2\pi}\int_{-\lambda}^{\lambda} \hat{K}_\lambda(y)G(iy)e^{iuy}dy. \quad (*)$$

The second term on RHS $\to 0$ as $u \to \infty$, by the Riemann-Lebesgue Lemma (I.6). Change variables $t \mapsto v$ by $u - t = v/\lambda$, and use $K_\lambda(v/\lambda) = \lambda K(v)$:

$$\lim_{u\to\infty}\int_{-\infty}^{\lambda u} \sigma(u - v/\lambda)K(v)dv = A\lim_{u\to\infty}\int_{-\infty}^{\lambda u} K(v)dv = A. \qquad //$$

Theorem (Wiener-Ikehara Theorem, additive form). If $S(t) = 0$ for $t < 0$, is non-decreasing and right-continuous, and the LST

$$f(z) := \int_{0-}^\infty e^{-zt}dS(t) = z\int_0^\infty S(t)e^{-zt}dt \qquad (z = x + iy)$$

exists for $Re\ z = x > 1$, and for some constant A the analytic function

$$g(z) = g(x+iy) := f(z) - \frac{A}{z-1} \to g(iy) \qquad (x \downarrow 1),$$

where $g(i.) \in L_1(-\lambda, \lambda)$ for each λ – then

$$e^{-t}S(t) \to A \qquad (t \to \infty).$$

Proof. Write $\sigma(t) := e^{-t}S(t)$. Then for $Re\ z = x > 0$,

$$F(z) := \hat{\sigma}(z) = \int_0^\infty S(t)e^{-(z+1)t}dt = f(z+1)/(z+1),$$

$$G(z) := F(z) - A/z = \frac{f(z+1)}{z+1} - \frac{A}{z} = \frac{g(z+1) - A}{z+1}$$

(definition of g: check).

As $S(t) = e^t \sigma(t) \uparrow$, $\sigma(w') \geq \sigma(w)e^{w'-w}$ if $w' \geq w$. So by the Proposition,

$$\begin{aligned} A &= \lim_{u\to\infty} \int_{-\infty}^{\lambda t} \sigma(u - v/\lambda)K(v)dv \\ &\geq \limsup_{u\to\infty} \int_{-a}^{a} \ldots \qquad (\sigma, K \geq 0) \\ &\geq \limsup_{u\to\infty} \sigma(u - a/\lambda)e^{-2a/\lambda} \int_{-a}^{a} K(v)dv, \end{aligned}$$

by the above inequality on σ. So

$$\limsup_{u\to\infty} \sigma(u) \leq \frac{e^{2a/\lambda}}{\int_{-a}^{a} K(v)dv}.A.$$

Take $a := \sqrt{\lambda}$:

$$\limsup \sigma(.) \leq \frac{e^{2/\sqrt{\lambda}}}{\int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} K}.A.$$

Let $\lambda \to \infty$:

$$\limsup \sigma(.) \leq A.$$

So σ ($\uparrow$) is bounded: $\sigma(.) \leq M$, say.

This gives an upper bound. for the lower bound, take $b > 0$. Now

$$K(v) := \frac{1 - \cos\ v}{\pi v^2} \leq \frac{2}{\pi v^2} \leq \frac{1}{v^2}.$$